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Lecture - 63
Fractional PDEs Part 3

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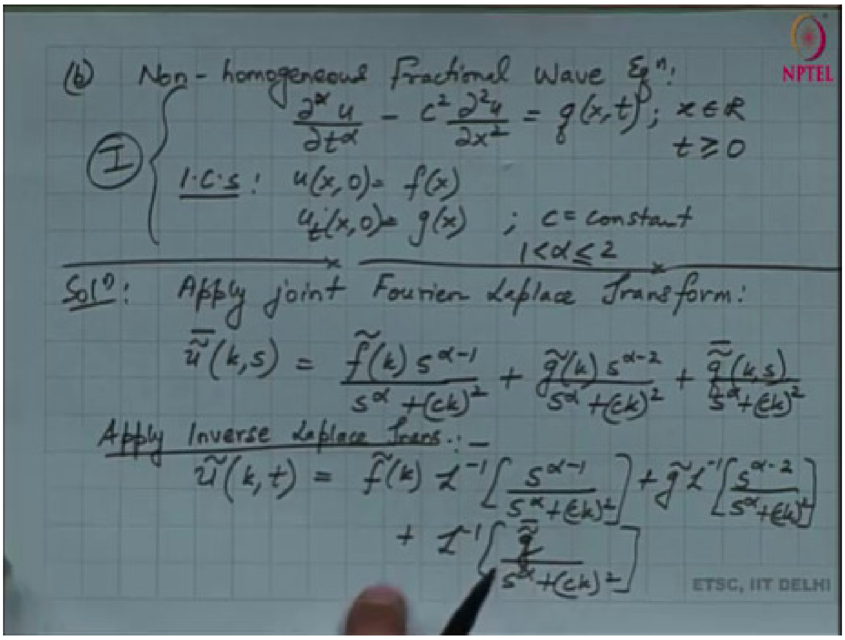

So, the second case I am going to talk about is the non homogeneous fractional wave equation. So, the non homogeneous fractional wave equation given by the following PDE:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - c^2 \frac{\partial^2 u}{\partial x^2} = q(x,t); \qquad x \in R, t \geq 0$$

I am also given the following initial conditions:

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x);$$
$$\text{c} = \text{constant}, \quad 1 < \alpha \leq 2$$

So, which means that if it is equal to 2 I get back my classical wave equation and that solution has already been discussed. So let us now call this as my I. So, if I were to apply joint Fourier Laplace transform, I get to see the following solution:

$$\bar{\tilde{u}}(k,s) = \frac{\tilde{f}(k)s^{\alpha-1}}{s^\alpha + ck^2} + \frac{\tilde{g}(k)s^{\alpha-2}}{s^\alpha + ck^2} + \frac{\bar{\tilde{g}}(k,s)}{s^\alpha + ck^2}$$

So, then let us apply start to apply the inverse transform. So, if I were to do that I am going to get solution in the Fourier space and the physical time. So, solution in the Fourier space and physical time is given by:

$$\tilde{u}(k,t) = \tilde{f}(k)\mathscr{L}^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha + ck^2}\right] + \tilde{g}\mathscr{L}^{-1}\left[\frac{s^{\alpha-2}}{s^\alpha + ck^2}\right] + \mathscr{L}^{-1}\left[\frac{\bar{\tilde{g}}}{s^\alpha + ck^2}\right]$$

So, I need to evaluate these three Laplace transform and I can see that I can evaluate the first two Laplace transform using Mittag Leffler expansions.

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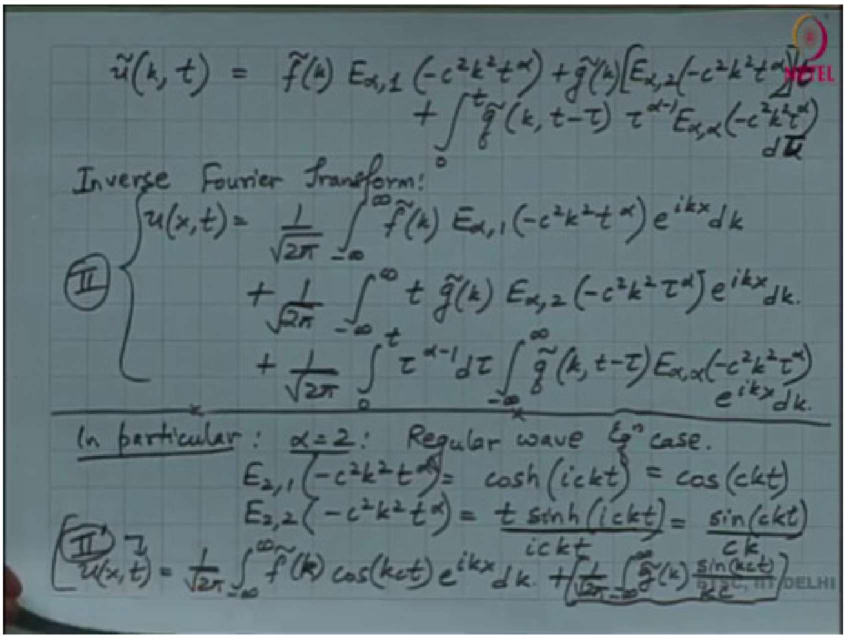

So, the solution in the Fourier transform and physical time space is:

$$\tilde{u}(k,t) = \tilde{f}(k)E_{\alpha,1}(-c^2k^2t^\alpha) + \tilde{g}(k)\big[E_{\alpha,2}(-c^2k^2t^\alpha)\big]t + \int_0^t \tilde{q}(k,t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}(-c^2k^2t^\alpha)d\tau$$

So, if I were to start applying the inverse Fourier transform the first two inverse transforms are quite straightforward in application. I see that my solution is:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)E_{\alpha,1}(-c^2k^2t^\alpha)e^{ikx}dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t\tilde{g}(k)E_{\alpha,2}(-c^2k^2t^\alpha)e^{ikx}dk$$
$$+\frac{1}{\sqrt{2\pi}}\int_0^t \tau^{\alpha-1}d\tau\int_{-\infty}^{\infty}\tilde{q}(k,t-\tau)E_{\alpha,\alpha}(-c^2k^2t^\alpha)e^{ikx}dk$$

So, I have three integrals to evaluate note that now from here onwards if I were to evaluate this I need to know the specific form of f, g and q, and once we know the specific form I can find the Fourier transform plug it into the integral and evaluate this inverse transform.

If we talk about a special case, the special case is when $\alpha = 2$, then the regular wave equation is :

$$E_{2,1}(-c^2k^2t^\alpha) = \cosh(ickt) = \cos(ckt)$$
$$E_{2,2}(-c^2k^2t^\alpha) = \frac{t\sinh(ickt)}{ickt} = \frac{\sin(ckt)}{ck}$$

So, , now, substituting both these expansion for $\alpha = 2$, I get that my solution is:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)\cos(kct)e^{ikx}dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(k)\frac{\sin(kct)}{kc}e^{ikx}dk$$
$$+\frac{1}{\sqrt{2\pi}}\frac{1}{c}\int_{-\infty}^{t}d\tau\int_{-\infty}^{\infty}\tilde{q}(k,\tau)\frac{\sin(kc(t-\tau))}{k}e^{ikx}dk$$

Let me call this as $II^{'}$ because this is a particular case.

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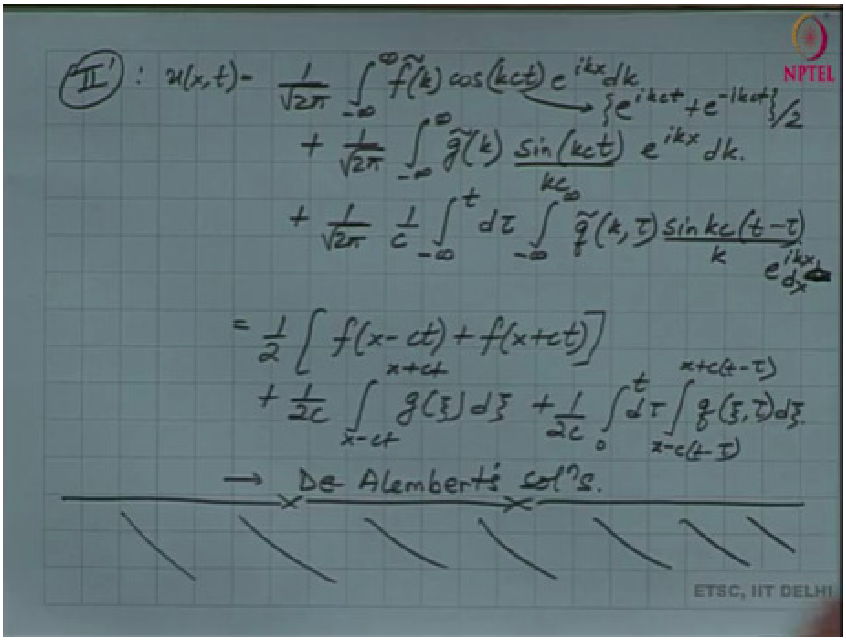

So, then note that in the first case if I were to replace my $\cos(kct)$ by $(e^{ikct} + e^{-ikct})/2$ and then we get,

$$= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)d\zeta + \frac{1}{2c}\int_{0}^{t} d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} q(\zeta,\tau)d\zeta$$

So, this is nothing but the well known De Alembert?s solution for the wave equation that we had found earlier for the regular wave equation. So, with this example I conclude our discussion on the fractional ODE?s; however, I continue my discussion on my fractional PDE?s namely we will see some special PDE?s in arising in fluids in signal processing and in quantum mechanics in my next lecture.