Integral Transform and Their Applications
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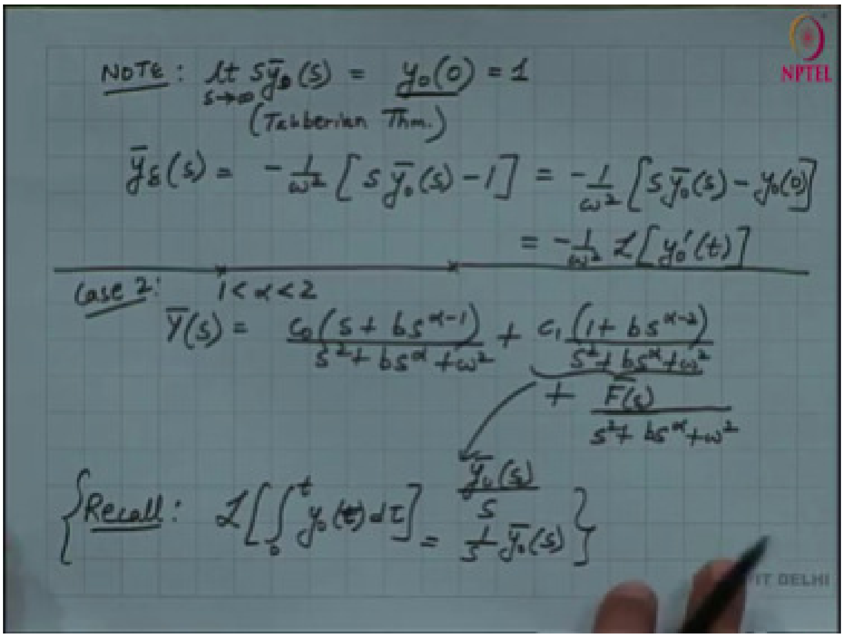
Lecture - 62
Fractional PDEs Part 2



So without discussing the solution to this 1st case I am going to discuss the 2nd case and then come back to the solution to both the cases. So, the case 2 is when $1 < \alpha < 2$ and in that case let me write down the transform solution. The transformed solution:

$$\bar{Y}(s) = \frac{c_0(s + bs^{\alpha-1})}{s^2 + bs^{\alpha} + \omega^2} + c_1\frac{(1 + bs^{\alpha-2})}{s^2 + bs^{\alpha} + \omega^2} + \frac{\bar{F}(s)}{s^2 + bs^{\alpha} + \omega^2}$$

where the second term is equivalent to $\frac{\bar{y}_0(s)}{s}$.

So, that is the expression if we go back and look at look in our case 1 which means I need to recall one more result for the Laplace transform. So, recall that the

$$\mathscr{L}\left[\int_0^t y_0(\tau)d\tau\right] = \frac{1}{s}\bar{y}_0(s)$$

So, if I were to evaluate the inverse transform of this expression this is going to be nothing but the integral of $y_0(t)$ in the inverse transform. So, let us now club the solution for both the cases to come to the answer for both the cases.

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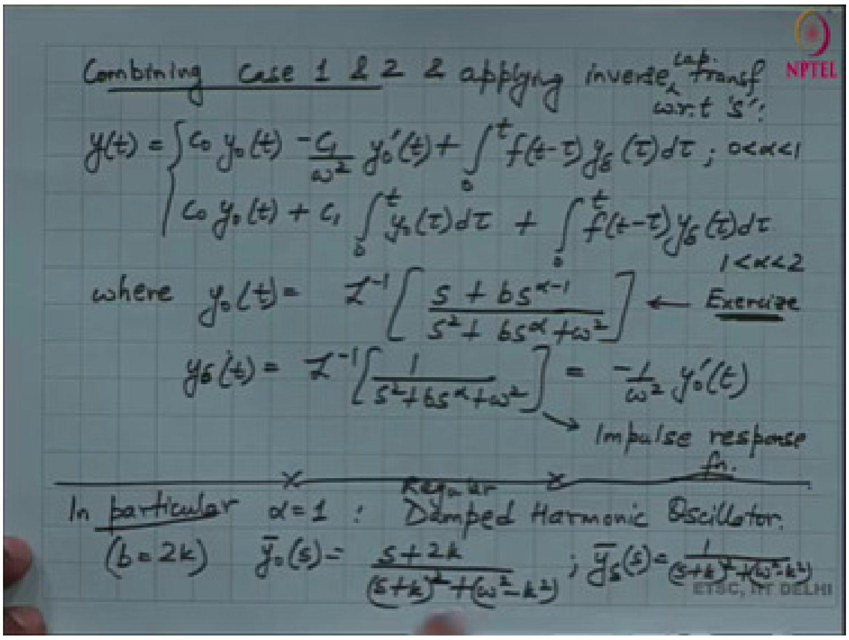

So, combining; cases 1 and 2 and applying inverse Laplace transform with respect to the variable s. So, then I get that the solution $y(t)$ is given by:

$$y(t) = \begin{cases} c_0 y_0(t) - \frac{c_1}{\omega^2} y_0^{'}(t) + \int_0^t f(t-\tau) y_\delta(\tau) d\tau, & \text{if } 0 < \alpha < 1. \\ c_0 y_0(t) + c_1 \int_0^t y_0(\tau) d\tau + \int_0^t f(t-\tau) y_\delta(\tau) d\tau, & \text{if } 1 < \alpha < 2. \end{cases}$$

where,

$$y_0(t) = \mathscr{L}^{-1}\left[\frac{s + bs^{\alpha-1}}{s^2 + bs^{\alpha} + \omega^2}\right]$$

$$y_\delta(t) = \mathscr{L}^{-1}\left[\frac{1}{s^2 + bs^{\alpha} + \omega^2}\right] = -\frac{1}{\omega^2} y_0^{'}(t)$$

We see that this was the denominator in our transformed plane or I can relate this in my regular ODE in the case of a signal processing example. So I can relate this denominator as my impulse response function.

So, this corresponds to my impulse response function in the ODE or the denominator in the transform plane. So, all I need to do is to evaluate this inverse and I got the solution to this problem. So, that I leave it as an exercise to the students. So, moving on let us look at more examples. So, before I do that let me just discuss some specific cases of this fractional simple harmonic oscillator. So, in particular; if I have $\alpha = 1$ then instead of alpha being a fraction, alpha is a derivative of order 1. So, then I am back to my regular damped harmonic oscillator. So If I assume $b = 2k$, then

$$\bar{y}_0(s) = \frac{s + 2k}{(s+k)^2 + (\omega^2 - k^2)},$$

$$\bar{y}_\delta(s) = \frac{1}{(s+k)^2 + (\omega^2 - k^2)}$$

So, we can quickly find the inverse transform of both these functions. So, let me just show you one particular case.

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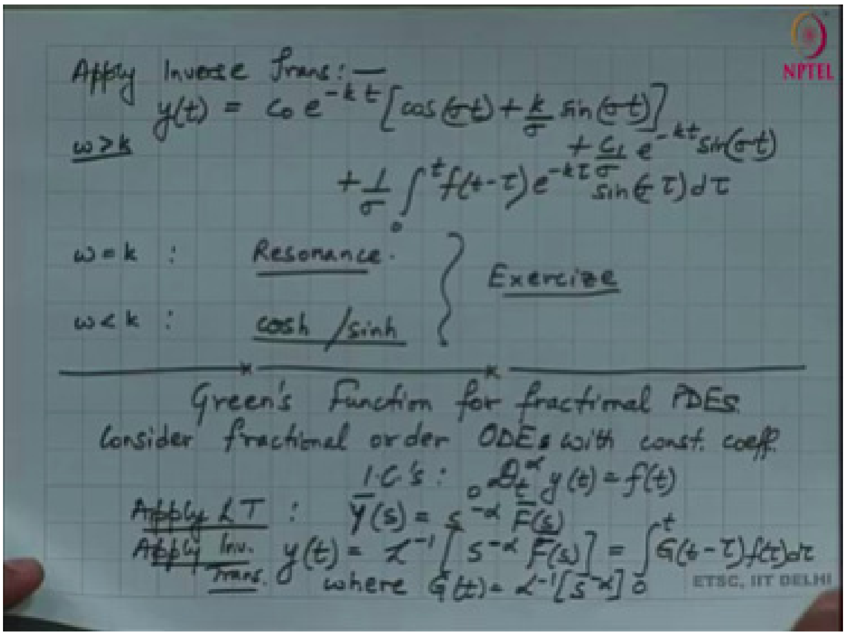

So, if I were to apply inverse transform, I get the following expression I get my solution $y(t)$. her, we will have three cases and let us discuss one of the case. For $\omega > k$,

$$y(t) = c_0 e^{-kt}\left[\cos(\sigma t) + \frac{k}{\sigma}\sin(\sigma t)\right] + \frac{c_1}{\sigma}e^{-kt}\sin(\sigma t) + \frac{1}{\sigma}\int_0^t f(t-\tau)e^{-k\tau}\sin(\sigma\tau)d\tau$$

Similarly I can find the other 2 cases $\omega = k$ and $\omega < k$. So, I see that $\omega = k$ will be the resonance case and $\omega < k$, we will get a solution in terms of cos hyperbolic and sine hyperbolic. So, I leave these 2 cases for the students to come to the expression. So, this is the case of a regular damped harmonic oscillator that is $\alpha = 1$.

So, let us move ahead to another example now. So, before that let me just introduce another concept known as the Green?s function for fractional PDE. So, before we start to solve equations which are partial differential equations the Green?s function is going to simplify our analysis and the solution to these equations.

So, let us see what are these Green?s function?

Before even going to PDE?s let us see what happens in the case of ODE?s. So, consider the following fractional order ODE?s with constant coefficient and I am given the initial conditions as follows:

$$\mathcal{D}_t^{\alpha} y(t) = f(t)$$

So, if I were to apply Laplace transform to these ODE?s, I am going to get:

$$\bar{y}(s) = s^{-\alpha}\bar{F}(s)$$

and then applying inverse Laplace transform, we get:

$$y(t) = \mathscr{L}^{-1}\left[s^{-\alpha}F(s)\right] = \int_0^t G(t-\tau)f(\tau)d\tau$$
$$\text{where,} G(t) = \mathscr{L}^{-1}\left[s^{-\alpha}\right]$$

Let me rewrite this expression now.

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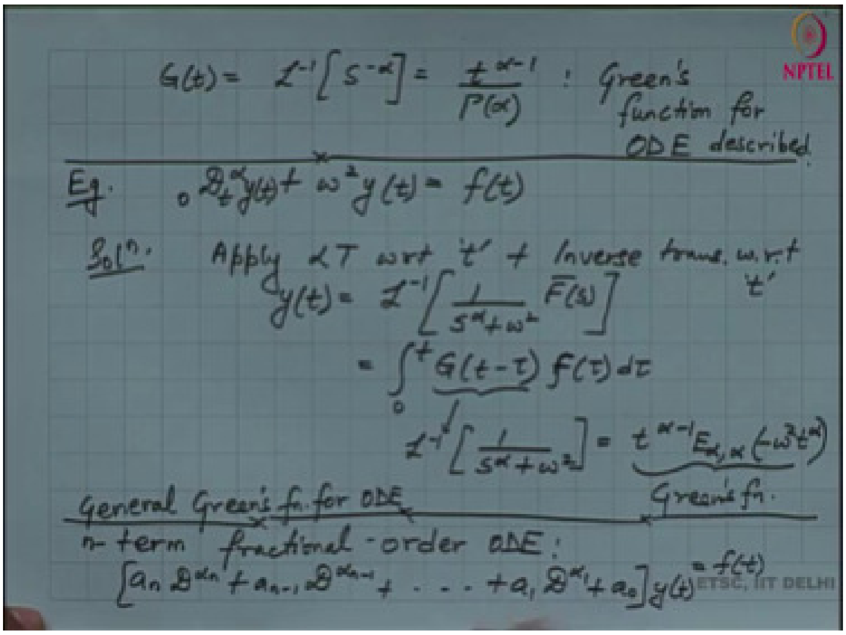

$$G(t) = \mathscr{L}^{-1}\left[s^{-\alpha}\right] = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

Note that I call this as my Green?s function for this ODE described. So, which means that whenever we have inverted a derivative operator or evaluating a fractional integral operator corresponding to that if we have an expression in the Laplace in the transform plane when we take the inverse of that, we are going to get the corresponding function known as the Green?s function.

So, let us now look at further examples. So now, I have another fractional order ODE
E.g.,

$${}_0\mathcal{D}_t^{\alpha} y(t) + \omega^2 y(t) = f(t)$$

Now so, if I were to find the solution to this problem, we will apply Laplace transform with respect to t and then apply inverse transform with respect to t to get my solution, i.e.,
Solution:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}+\omega^2}\bar{F}(s)\right] \\ &= \int_0^t G(t-\tau) f(\tau) d\tau \end{aligned}$$

So, I have just kept some of the steps to come to right away to the inverse transform of the right hand side. So, I see that my Green?s function here $G(t)$ will be the Laplace inverse transform of this following expression here:

$$\mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}+\omega^2}\right] = t^{\alpha-1} E_{\alpha,\alpha}(-\omega^2 t^{\alpha})$$

due to the use of Mittag Leffler expansion. So, these are my Mittag Leffler expansion with factors alpha and beta both equal. So, these are my Green?s function for this particular ODE. Note that the Green?s function is coming out from the operator equivalent of the ODE. So, we see that the operator here in this particular equation was $(\mathcal{D}^{\alpha} + \omega^2)$ and hence the Green?s function was the Laplace transform inverse of the corresponding operator equivalent of this ODE.

So, we can always generalize this Green?s function method. So, for an n term, this is the general description of the Green?s function.
If I have an n term fractional order ODE given by the following:

$$\left[a_n \mathcal{D}^{\alpha_n} + a_{n-1}\mathcal{D}^{\alpha_n - 1} + ... + a_1 \mathcal{D}^{\alpha_1} + a_0\right] y(t) = f(t)$$

So, if I were to find what is the Green?s function of this nth order ODE, all I need to do is look at this operator which is operating on the solution and find the inverse transform of the corresponding Laplace transform of this operator.

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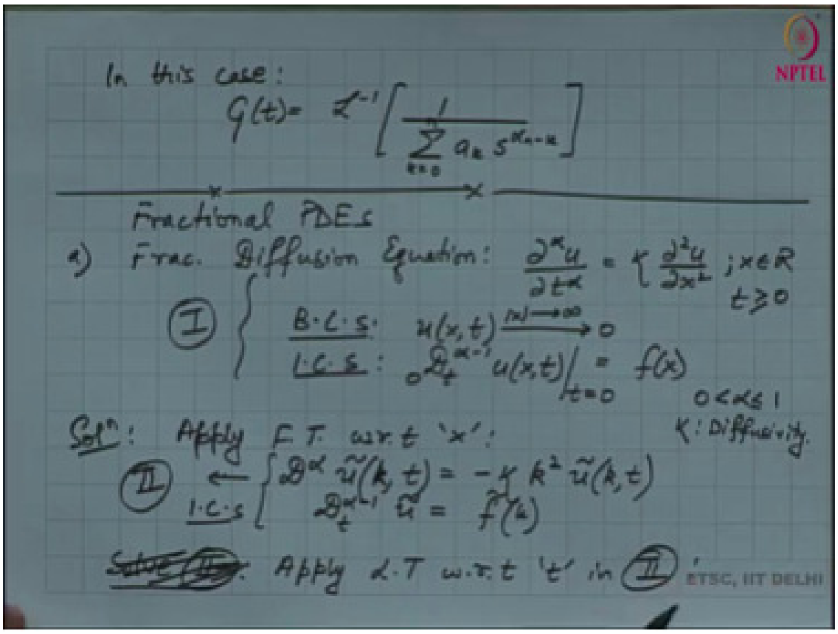

So, what I mean by this is in this particular case is

$$G(t) = \mathscr{L}^{-1}\left[\frac{1}{\sum_{k=0}^{n} a_k s^{\alpha_n - k}}\right]$$

Let us now continue our discussion and we are going to use our method of Green?s function and try to solve certain fractional PDE?s. So, let us look at some particular cases of fractional PDE?s. So, I am going to start my discussion here by looking at the fractional diffusion equation. So, fractional diffusion equation given by the fractional derivative:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2}; \quad x \in R, t > 0$$

So, then I must also give the boundary conditions. The boundary conditions are given by

$$u(x,t) \longrightarrow 0 \quad \text{as} \quad |x| \to \infty$$

and then I have that the initial conditions are given by the following set of the conditions:

$${}_0\mathcal{D}_t^{\alpha-1} u(x,t) = f(x) \quad \text{at} \quad t = 0$$

where, $0 < \alpha \leq 1$ and then Kappa is the diffusivity or the diffusion constant. So now we see that x varies from negative infinity to infinity. So, f will be a choice of my Fourier transform while t is non negative t will be the choice of my Laplace transform. So, let me first apply Fourier transform i.e. Fourier transform with respect to the variable x here. So, if I were to do that we see that the term on the right hand side of this equation the diffusion equation is a derivative with respect to time and then I see that coming back to this equation I see that the left hand side is the

$$\mathcal{D}^{\alpha}\tilde{u}(k,t) = -\kappa k^2 \tilde{u}(k,t)$$

So, then let me also look at the initial condition and apply the Fourier transform of the initial condition is given by

$$\mathcal{D}_t^{\alpha-1}\tilde{u} = \tilde{f}(\kappa)$$

I see that if I were to solve let me call the original equation as my I and let me call this as II. So, if I were to solve my ODE given by II, well this is the ODE given by 2 is partly in the transform plane and partly in the physical plane. So, let me further move ahead and apply Laplace transform with respect to the variable t in my set of equations given by II. So, when I do that I am going to use my Laplace transform of the fractional derivative, where $0 < \alpha < 1$.

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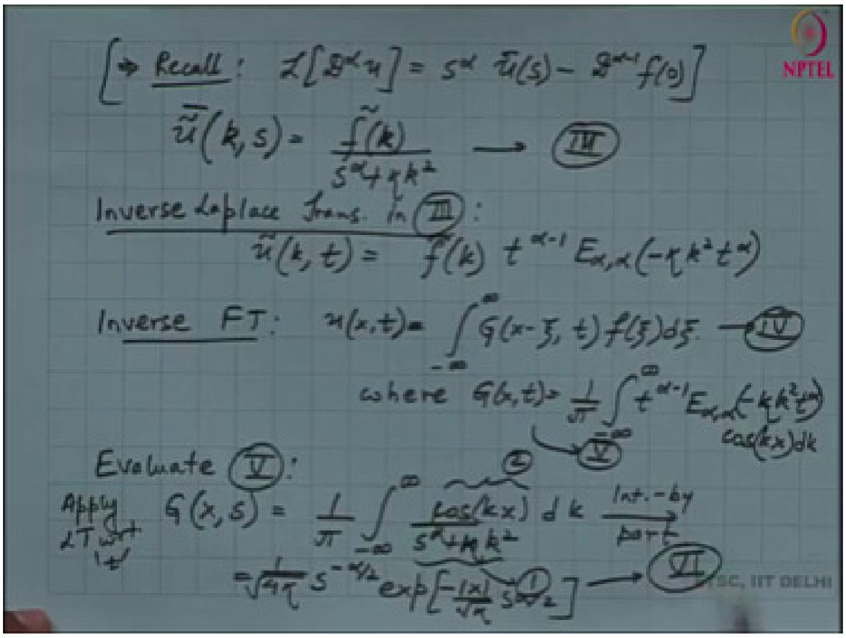

So, I am recalling the following:

$$\mathscr{L}\big[\mathcal{D}^{\alpha}u\big] = s^{\alpha}\bar{u}(s) - \mathcal{D}^{\alpha-1}f(0)$$

So, I am applying joint transform, which means my solution is in the space of Fourier Laplace transform. So, my joint Fourier Laplace transform with respect to the transformed variable (k, s) is

$$\bar{\tilde{u}}(k,s) = \frac{\tilde{f}(k)}{s^{\alpha} + \kappa k^2}$$

Let me call this expression as my expression III. So, this is after the application of both Fourier and Laplace transform and using my initial condition which was transformed in the Fourier plane. So, then if I were to apply the inverse Laplace transform in III, I get the solution in the transformed plane with respect to space, but the physical plane with respect to time as follows:

$$\tilde{u}(k,t) = \tilde{f}(k)t^{\alpha-1}E_{\alpha,\alpha}(-\kappa k^2 t^{\alpha})$$

So, again this is the Mittag Leffler expansion. So, then if I were to apply my inverse Fourier transform I see the following:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\zeta,t) f(\zeta) d\zeta$$

where,

$$G(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\kappa k^2 t^\alpha) \cos(kx) dk$$

Note that we have certain symmetry in this problem. The symmetry is with respect to k here if I replace well the symmetries with respect to the space, if I replace k 1 by minus k the solution does not change. So, which means instead of the regular Fourier integral I am going to use a cosine integral because of the symmetry in the problem.

So, my Green's function is the inversion of this function via this integral. Now one more step that is possible and that is to evaluate this Green's function and once I evaluate I put it back in this expression IV to come to my solution. So, if I were to evaluate let us evaluate this expression V. So, if I were to evaluate my expression V my Green's function. So, to evaluate V we see that there is no direct method of evaluation of this integral, but to do that let us first take the Laplace transform of V to see what happens. So, apply Laplace transform with respect to my variable t. So, in that case,

$$G(x,s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(kx)}{s^\alpha + \kappa k^2} dk$$

So, note that this particular integral can be solved by integration by parts. So, I can use integration by parts to integrate this expression. Notably if I choose this numerator as my second function and my denominator here as my first function I see that I get this Green's function:

$$= \frac{1}{\sqrt{4\kappa}} s^{-\alpha/2} exp\left[\frac{-|x|}{\sqrt{\kappa}} s^{\alpha/2}\right]$$

So, I get that the Green's function is the above expression here and that comes right away by integration by parts. Now this is the Laplace transform of the Green's function which means I have to do one more step. So, let me call this as my step number VI. So, if I were to now take the inverse transform of this step number VI.

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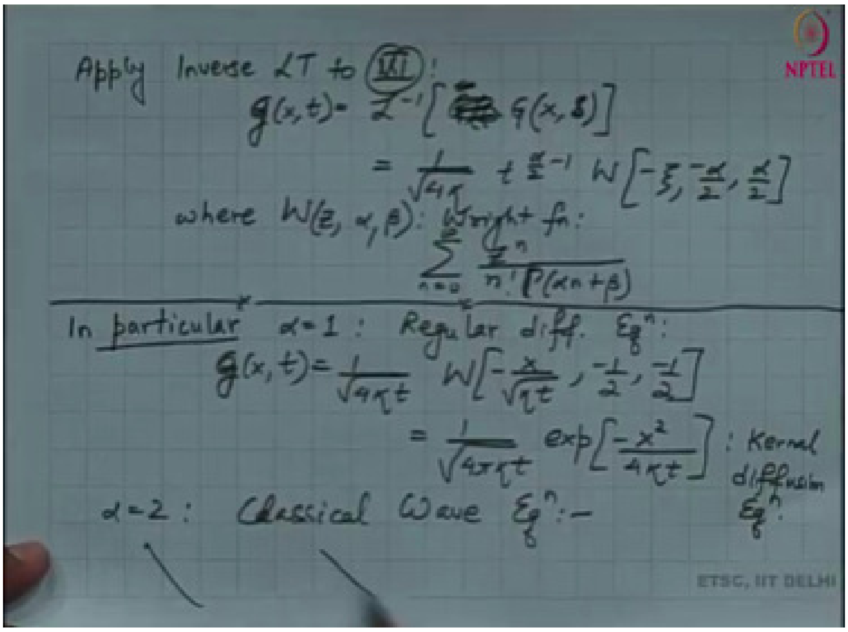

So, apply inverse Laplace transform to step number VI, I come to my Green?s function:

$$g(x,t) = \mathscr{L}^{-1}\big[G(x,s)\big]$$
$$= \frac{1}{\sqrt{4\kappa}} t^{\frac{\alpha}{2}-1} W\left[-\zeta, -\frac{\alpha}{2}, \frac{\alpha}{2}\right]$$

where, the write function is:

$$W(z,\alpha,\beta) = \sum_{n=0}^{\infty} \frac{z^n}{n!\ \Gamma(\alpha n+\beta)}$$

So, we see that the solution is found by plugging in this write function into my step number IV to find the solution to the fractional diffusion equation. So, let me just end this discussion and move ahead by giving you a particular case. So, in particular if I choose my $\alpha = 1$.—

So, this is my regular diffusion equation.Then my Green's function is:

$$g(x,t) = \frac{1}{\sqrt{4\kappa t}} W\left[-\frac{x}{\sqrt{\kappa t}}, -1/2, 1/2\right]$$
$$= \frac{1}{\sqrt{4\kappa t}} exp\left[-\frac{x^2}{4\kappa t}\right]$$

So, we see that this is the classic kernel for the diffusion equation. So, I get back the regular case by plugging in $\alpha = 1$ and then if I plug $\alpha = 2$, I am going to get my classical wave equation. So, let us now discuss the solution to the fractional wave equation. So, let us look at another case now.