**Integral Transforms and Their Applications**
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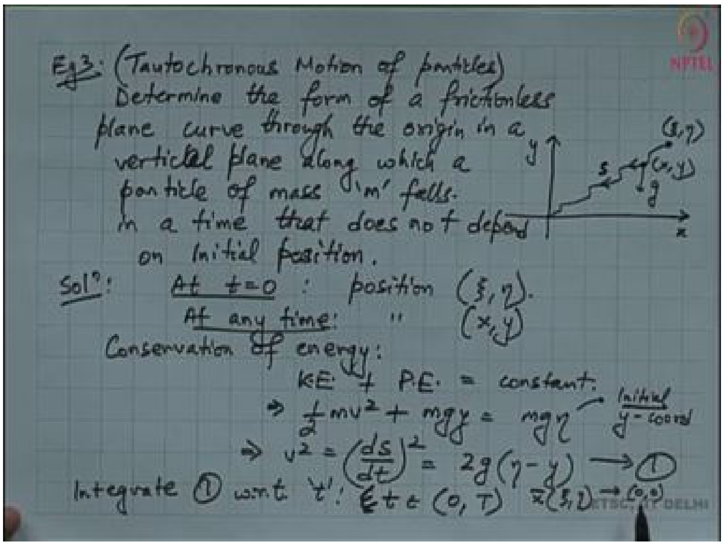
**Lecture –60**
**Fractional ODEs, Abel's Integral Equations Part 3**

Let me show you some other examples specially; let me show you some word problems. So, this question says that suppose so, the problem say suppose in a 2 D plane let me just draw the diagram. So, suppose in a 2 D plane I have a particle, let me say that the particle is located at some point $\zeta, \eta$ and the particle is under the action of gravity.

So, the particle is under the action of gravity and it is going to and due to the action of gravity it is going to fall. So, due to the action of gravity it will fall, let me call that that falling trajectory by $x, z$ . So, the question says what is the curve such that the particle falls under the action of gravity and it this particle falls in such a way that this trajectory is independent of my initial location. So, I need to find this curve this trajectory of the particle going from let us say some point to the origin completely under the action of gravity such that this curve is independent of the initial condition.

So, let me write down the word problem to this question. So, this is also known as the famous Tautochronous motion of particles; so, tautochronous motion of particles. So, this question is to determine the form to determine the form of a friction-less to determine the form of a friction-less plane curve through the origin in a vertical plane along which a particle of mass m falls; in a time that does not does not depend on initial position.

So, we need to find the trajectory along in which the particle is falling purely under the action of gravity such that the trajectory does not depend on the initial position of the particle . So, then well for the sake of starting the problem let me just say that at $t = 0$ the particle is at position $\zeta, \eta$. So now, let me just say that at any time, the position is denoted by this these two variable x and y. So, that is what highlighted in this figure here.




So, now I know that by conservation of energy of the particle I know that the kinetic energy the sum total of all the kinetic energy plus the potential energy of the particle is a constant.

So, the total energy of the particle is conserved. So, which means that let me write down the kinetic energy; this is,

$$K.E + P.E = \text{ constant}$$
$$\Rightarrow \frac{1}{2}mv^2 + mgy = mg\eta$$

So, that is equal to a constant. So, well so, $\eta$ denotes the y coordinate of the particle. So, this is the initial y coordinate of the particle. So, that is a constant here which means that I have the following I have $v^2$ so, let me rewrite this expression in terms of $v^2$. So, $v^2$ this is also equal to ,

$$\Rightarrow v^2 = \left(\frac{ds}{dt}\right)^2 = 2g(\eta - y) \quad ...(1)$$

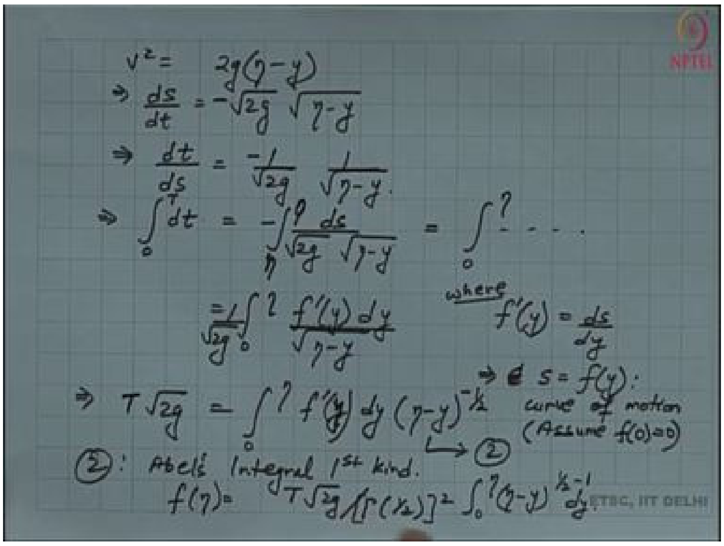
So, I have re-written this energy conservation to find v in terms of y. So, then let me just integrate 1 with respect to the time t.

So, to do that my time my time variable is from 0 to let us say time point capital T. So, which means I need to find this capital T, this time point T where the particle x goes from $\zeta, \eta$ to the origin.

So, at $t = 0$ that is the initial position of the particle and at $t = T$ the final position is the origin $(0, 0)$. So, then let us integrate (1) so, when I integrate (1) so, I have the following:

$$v^2 = 2g(\eta - y)$$

$$\Rightarrow \frac{ds}{dt} = -\sqrt{2g}\sqrt{\eta - y}$$




$$\Rightarrow \frac{dt}{ds} = \frac{-1}{\sqrt{2g}}\frac{1}{\sqrt{\eta - y}}$$

$$\Rightarrow \int_0^T dt = -\int_\eta^0 \frac{ds}{\sqrt{2g}\sqrt{(\eta - y)}}$$

$$= \frac{1}{\sqrt{2g}} \int_0^{\eta} \frac{f'(y)dy}{\sqrt{\eta - y}}$$

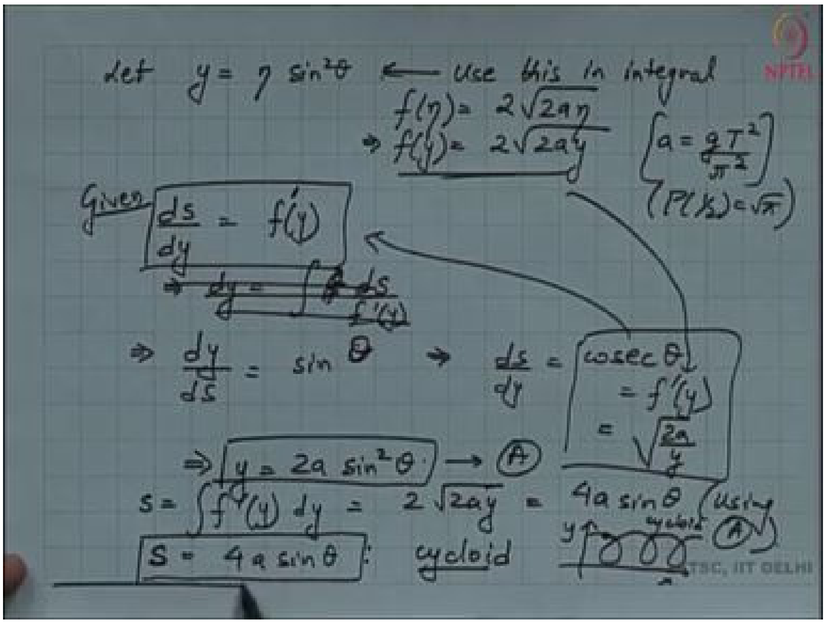
Where,

$$f'(y) = \frac{ds}{dy} \quad \text{and} \quad s = f(y)$$

$$T\sqrt{2g} = \int_0^{\eta} f'(y)dy(\eta - y)^{-\frac{1}{2}} \qquad .....(2)$$

Here , (2) is Abel's Integral 1st Kind,

$$f(\eta) = \frac{T\sqrt{2g}}{[\Gamma(\frac{1}{2})]^2} \int_0^{\eta} (\eta - y)^{\frac{1}{2}-1} dy$$




$$\text{Let,} \quad y = \eta \sin^2 \theta$$

Use this in Integral,

$$f(\eta) = 2\sqrt{2a\eta}$$
$$f(y) = 2\sqrt{2ay}$$

Here,

$$a = \tfrac{gT^2}{\pi^2}$$
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Given,

$$\frac{ds}{dy} = f'(y)$$

$$\Rightarrow \frac{dy}{ds} = \sin\theta$$

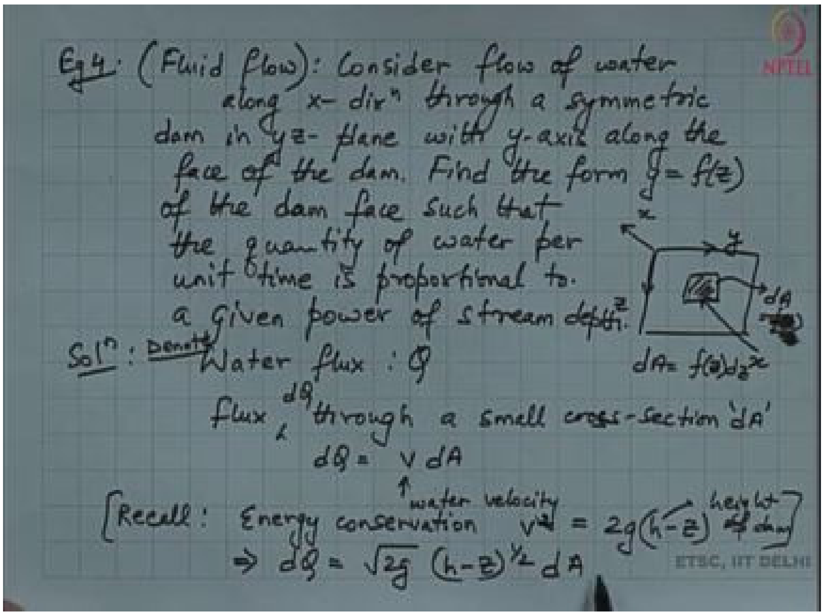
$$\frac{ds}{dy} = \operatorname{cosec}\theta = f'(y) = \sqrt{\frac{2a}{y}}$$

$$y = 2a\sin^2\theta \qquad ....(A)$$

$$s = \int f'(y)dy = 2\sqrt{2ay} = 4a\sin\theta$$

$$S = 4a\sin\theta$$

So, using my expression A here ,this is my standard cycloid . So, what it shows is that this particular particle is going to follow a cycloid motion. So, what is the cycloid motion? If people who drive cycles who drive cycles we know that the tyre of a cycle, if we were to track a point on the tyre of a cycle it is going to; if it is going to go ahead in a helical fashion and that is the motion of the cycloid. So, that is my cycloid ok. So, that is the expression that is given by the by the trajectory of the particle falling purely under the action of gravity. So, let us now consider another example.




So, let me show you another example. This example is in the area of fluid flow and again Abel's integration equation is quite useful in solving this problem. So, the problem says consider the flow of water along x -direction through a symmetric through symmetric dam; consider the flow of water in the x -direction through a symmetric dam in the y-z plane with y-axis along the face with y-axis along the face of the dam. So, let me just draw the figure. So, the figure is as follows. (Refer the above slide)

So, let me say that I have a dam. So, it is a 2 D dam and in such a way that the y axis is along the face of the dam. So, which means and the flow of the water is along the x-direction. So, let me call that the flow of the water is as follows. So, this is my x-direction. So, this is my x axis here; so, which means my z- axis is perpendicular to y shown here as follows (Refer the aove slide)

So, then with y-axis along the face of the dam and I have to find the form of y equal to f z the face of the dam. So, find the form of this face of the dam of the dam such that the quantity of water per unit time per unit time is proportional to a given power of stream depth.

So, the problem says that I have to find what is the shape of this particular dam such that the water is flowing perpendicular to the dam. And, such that the quantity of the water per unit time that is flowing through the dam is proportional to certain given power of the depth which means that the water which is flowing let us say let me denote the water quantity by q and q will be a function of this dam depth which is z or a power f z ($^z$).

Solution:

So, power of this height of the dam let us say the height is h or z. So, let me start this problem. So, I am given let us call let me say that the water flux. So, let me denote let me denote the water flux Q; so, let me denote the water flux by Q. So, I am going to find the flux the flux through a small cross section of the dam, the flux through a small cross section the flux through the small cross section dA of the dam.

So, we see that flux through the small cross section flux dQ through the small cross section d of the dam is given by dQ,

$$dQ = VdA$$

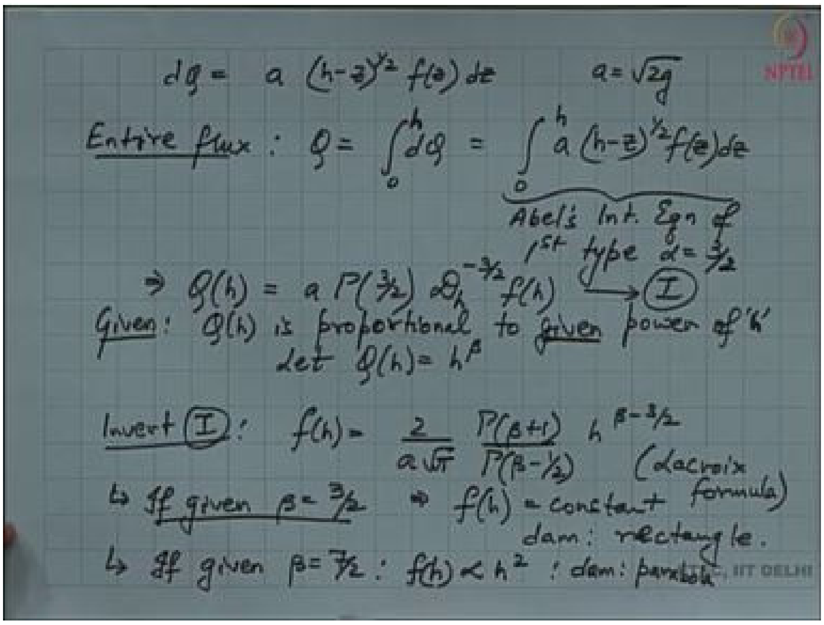
So, I see that this now I am going to write this expression of v let me recall again energy conservation recall energy conservation principle that is the energy of the water the energy of the water at any given height will be will be conserved. So, which means that I have the velocity $v^2$ of the water is,

$$v^2 = 2g(h - z)$$

$$\Rightarrow dQ = \sqrt{2g}(h - z)^{1/2} dA$$

So, then let us now re-write this expression for this small flux to come to a new expression.

$$dQ = a(h - z)^{1/2} f(z) dz \qquad \text{where,} \quad a = \sqrt{(2g)}$$



Entire Flux:

$$Q = \int_0^h dQ = \int_0^h a(h-z)^{1/2} f(z) dz$$

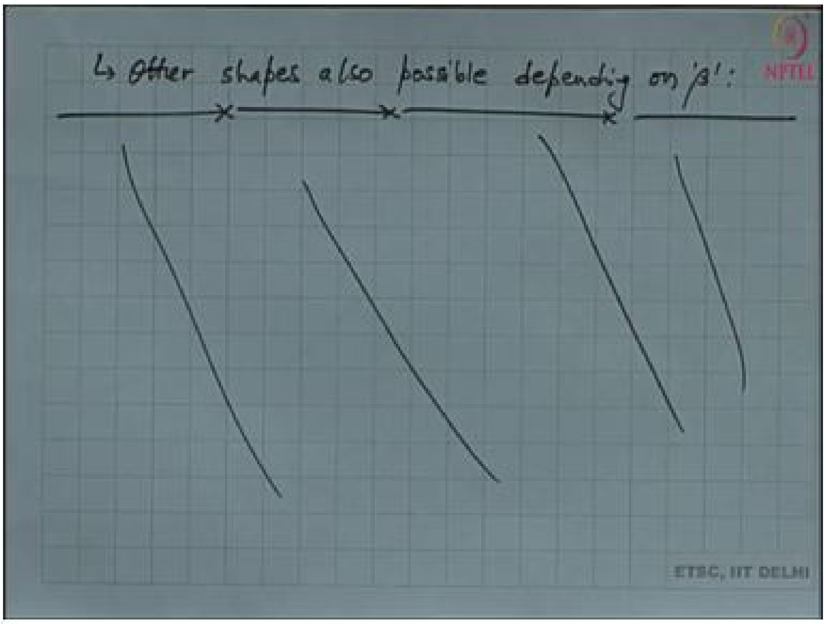
This is the Abel's Integrals equation of type 1 ($\alpha = \frac{3}{2}$,

$$Q(h) = a\Gamma(3/2) D_h^{-3/2} f(h) \qquad ....(I)$$

Given,Q(h) is proportional to given power of h, Let, $Q(h) = h^{\beta}$,
Now Invert the equation (I),

$$f(h) = \frac{2}{a\sqrt{\pi}} \frac{\Gamma(\beta+1)}{\Gamma(\beta-1/2)} h^{\beta-3/2} \qquad ;\text{Lacroix Formula}$$

If given $\beta = \frac{3}{2}$,
then, f(h)= constant and dam=rectangular.
If given $\beta = \frac{7}{2}$,
then, f(h) is proportional to $h^2$ and dam=parabola.




So, which means that we can similarly proceed in similar fashion to come to other forms of the expression for the shape of the dam.

So, which means that which means that other shapes are also possible depending on the value of beta that is provided. So, that is going to complete our discussion on the problem and that is also going to complete our discussion on this, this particular lecture. In my next lecture I am going to continue my discussion on fractional ODE's moving onto fractional PDE's. And, I am going to show you some specific cases of fractional PDE's namely in simple harmonic oscillations in fluid flow and in signal processing. So, thank you for listening. Thank you very much.