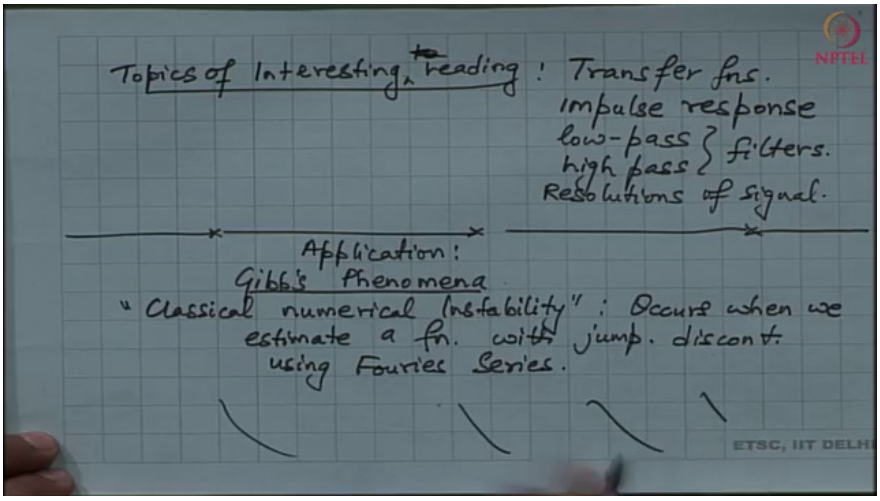
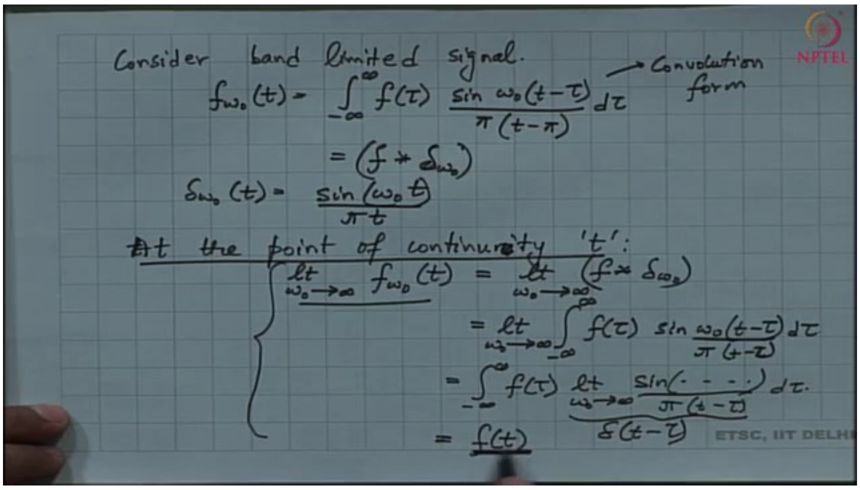
Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture – 2
Introduction to Fourier Transforms Part - 03

So, what we see is that this sort of a representation of a discontinuous function by a series of continuous Fourier transform functions leads to some oscillations or wiggling in the representation of the function. So, let me now highlight this case study with an example.



So, as I said Gibbs phenomena is quite common in classical numerics so, this is a classical numerical instability. It occurs when we have to estimate a function with a jump discontinuity. So, suppose you have a discontinuous function and you are representing it with a Fourier transform, then we are going to see most likely we are going to see Gibbs phenomena. So, let me give you an example; so, let me give me an example.



So, consider this band limited signal:

$$f_{w_o}(t) = \int_{-\infty}^{\infty} f(\tau) \frac{\sin \omega_0 (t-\tau)}{\pi(t-\pi)} d\tau$$

So, this is in the convolution form; convolution form of representing the signal ok. So, well so, this is what this is nothing,

$$= (f * \delta_{w_o})$$

$$where, \delta_{w_o}(t) = \frac{\sin(\omega_0 t)}{\pi t}$$

So, as we can see that as ω goes to omega 0 goes to infinity this particular definition goes to the actual definition of delta function.

So, if I have t so, at the point of continuity. So, suppose the function is continuous continuity right. So, at the point of continuity I have that if I were to take the limit omega going to infinity of this function the signal is given as:

$$\lim_{\omega_o \to \infty} f_{\omega_0}(t) = \lim_{\omega_o \to \infty} (f * \delta_{w_o})$$

$$= \lim_{\omega_o \to \infty} \int_{-\infty}^{\infty} f(\tau) \sin \frac{\omega_0(t-\tau)}{\pi(t-\tau)} d\tau$$

And then we see that if I were to take this limit inside the integral, I get:

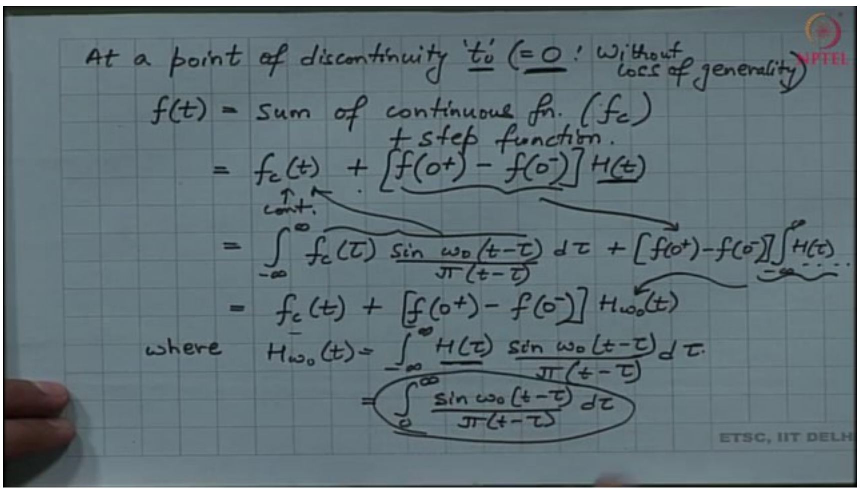
$$= \int_{-\infty}^{\infty} f(\tau) \lim_{\omega_o \to \infty} \sin \frac{\omega_0(t-\tau)}{\pi(t-\tau)} d\tau$$

$$= f(t)$$

We see that this particular limit is my delta function. So, this is

$$\lim_{\omega_o \to \infty} \sin \frac{\omega_0(t-\tau)}{\pi(t-\tau)} = \delta(t-\tau)$$

So, what is the conclusion? The conclusion is at the point of continuity if at this point t, if it is if the function is continuous I get that this limit is equal to the value of the function, the limit is equal to the value of the function.



Now, let us look at; let us look at another point which is the point of discontinuity. So, at the point of discontinuity let us say t 0. So, I am going to t take t 0 to be 0 without loss of generality right. So, let me just take t 0 to be a special point let us say it is 0. If it is not 0 then you can always shift the function, so, that the point of discontinuity lies at 0 right.

So, then in that case I can write my function f as a sum of continuous function let us say I call this as as f_c plus a step function right. So, let me show you how.

$$f(t) = f_c(t) + \left[f(0^+) - f\left(0^-\right)\right] H(t)$$

So, my discontinuity can be represented as f at 0 plus minus f at 0 minus. So, that is where the discontinuity is times the heavy side function t right.

So, the heavy side function becomes non becomes one when t is greater than 0 otherwise it is this discontinuity disappears right. So, then what do I have, I have the following:

$$= \int_{-\infty}^{\infty} f_c(\tau) \frac{\sin w_0(t-\tau)}{\pi(t-\tau)} d\tau + [f(0^+) - f(0^-)] \int_{-\infty}^{\infty} H(\tau)..$$

So, I have this function which is a function of time and the others which are constant I have taken it out of the integral right. So, then of course, I am left with well what I have is that of course, this is equal to f_c I saw that this convolution reduces back to this function in case of in case the function is continuous right:

$$f(t) = f_c(t) + \left[f(0^+) - f\left(0^-\right)\right] H_{\omega o}(t)$$

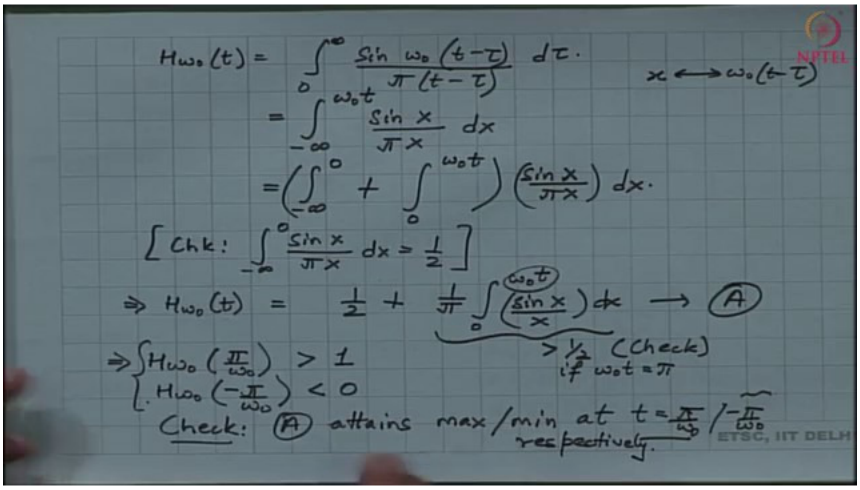
So, I am describing this integral by:

$$H_{\omega_o} = \int_{-\infty}^{\infty} H(\tau) \frac{\sin \omega_0(t-\tau)}{\pi(t-\tau)} d\tau$$

So, where my $H_{\omega_o}(t)$ is,

$$= \int_{0}^{\infty} \frac{\sin \omega_0(t-\tau)}{\pi(t-\tau)} d\tau$$

So, as this is a heavy side function this reduces to 0 to infinity sin omega t minus tau divided by pi t minus tau d tau. So, then what I have is, that if I where to replace. So, let us now look at this integral.



So, this integral so, my $H_{\omega_o}(t)$ is again let me write it again so, this is from,

$$H\omega_0(t) = \int_0^{\infty} \frac{\sin \omega_0(t-\tau)}{\pi(t-\tau)} d\tau$$

$$x \leftrightarrow \omega_0(t-\tau)$$

$$= \int_{-\infty}^{\omega_o t} \frac{\sin x}{\pi x} dx$$

$$= \left(\int_{-\infty}^{0} + \int_{0}^{\omega_o t} \right) \left(\frac{\sin x}{\pi x} \right) dx$$

$$\left[\text{chk: } \int_{-\infty}^{0} \frac{\sin x}{\pi x} dx = \frac{1}{2} \right]$$

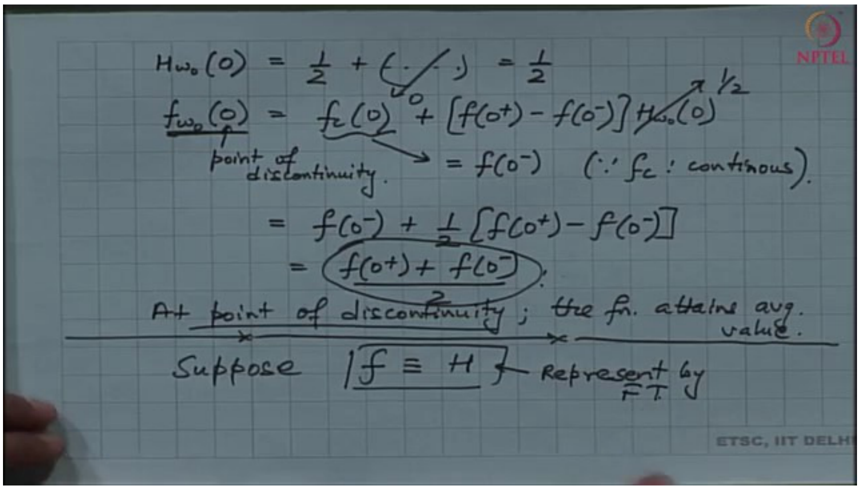
$$H_{w_0}(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\omega_o t} \frac{sinx}{x} dx$$

So, what I have is now you can also check that this particular function this thing is greater than half. So, you need so, students need to check that as well. So, this is greater than half; if $\omega_o t$ is equal to π then second term in above equation will be greater than $\frac{1}{2}$

$$\begin{cases} H\omega_0 \left(\frac{\pi}{\omega_0} \right) > 1 \\ H_{\omega_0} \left(-\frac{\pi}{\omega_0} \right) < 0 \end{cases}$$

check the above equation attains max/min at $t = \frac{\pi}{\omega_o} / -\frac{\pi}{\omega_o}$ respectively.

So, what I am trying to say here is, if you were to evaluate the maximum and minimum of this function, the maximum is at this time point and the minimum is at this time point respectively.



Which means that H_ω at t equal to 0 right, this is identically equal to half because the second integral that vanishes right. So, H_{ω_o} at t equals to 0 is half right. And of course, what I have is that if I my function f of omega 0 evaluated at time point 0 so, this is my point of discontinuity if you recall. So, this is my point of discontinuity.

$$H_{w_0}(0) = \frac{1}{2} + 0 = \frac{1}{2}$$

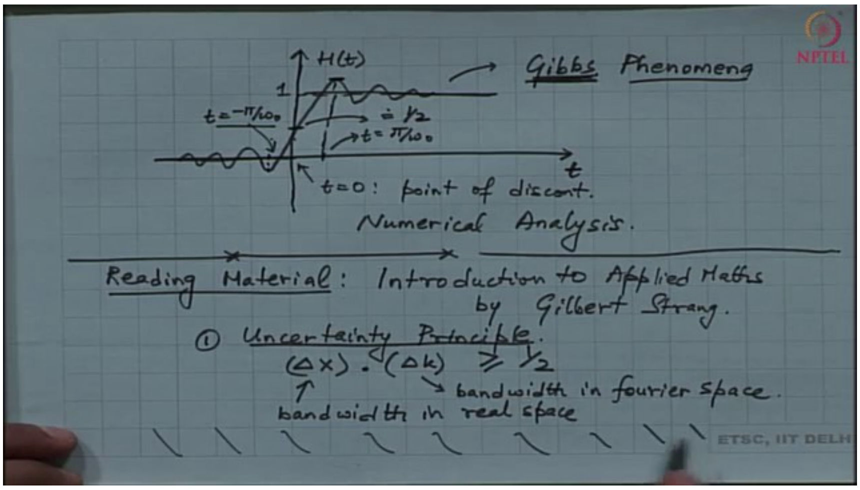
$$f_{\omega_o}(0) = f_c(0)^+ \left[f\left(0^+\right) - f\left(0^-\right)\right] H_{\omega_o}$$

$$= f(0^-) + \frac{1}{2}\left[f\left(0^+\right) - f(0^-)\right]$$

$$= \frac{f\left(0^+\right) + f\left(0^-\right)}{2}$$

So, what I say what I show here is that at point of discontinuity at point of discontinuity the function attains the average value.

So, you can see that this is the average value of the function at the point of discontinuity. So, what have I shown so far? What I have shown is the following. So, suppose my function f is identically equal to my heavy side function right. So, if I were to represent this function using Fourier transform represent so, let me just separate this part. So, represent this by Fourier transform.



(Refer above slide for below part) So, what have we got here. So, let us see that. So, what I have here is the following. So, I on my x axis is my variable t and on my y axis is my variable H(t) which is 1 for t positive and 0 for t negative. So, t equal to 0 is the point of discontinuity right. So, if I were to replace this by a Fourier transform of this function, we are going to get the Fourier transform which looks like this; it looks like this this and so on right. Where this maximum value this maximum value is at time point $t = \frac{\pi}{\omega_o}$, this minimum value is at $t = \frac{-\pi}{\omega_o}$ right. And as we proceed further, this t increases that is as omega increases as omega increases sorry omega decreases. The time increases and this periodicity this wiggling it starts to decay, but the maximum value is at this value $t = \frac{\pi}{\omega_o}$. The minimum value is at this value and at 0 I get that this is the function is equal to half. So, it attains the half of this value plus this value right.

So, this is a common scenario that we see and also term it as the Gibbs phenomena ok. So, this has quite a lot of application in numerical analysis right people doing computing, people doing you know numerical discretization of complex codes they are frequently going to incur the Gibbs phenomena.

Now, finally, I want to also mention some other reading material. So, this is also given in my original introduction or my introduction hand out, please refer to more topics related to a so, look at this book Introduction to Applied Maths by Gilbert Strang. And one very I interesting topic to read is the s, called uncertainty principle.

So, what does uncertainty principle says is that suppose we have the band width; suppose we have the bandwidth in real space. So, this is my band width; band width in real space and I have the band width in. So, this is my band width in Fourier space.

The uncertainty principle says commonly known as the Heisenberg uncertainty principle it says that this times the product of these two bandwidths cannot be lower than half right. So, what means is that suppose you have accurately resolved your physical space then this principle tells me that there is an in accuracy in the Fourier space and vice versa. So, we cannot accurately resolve both the physical and the Fourier space simultaneously ok. So, I end this lecture at this point in the next lecture I am going to introduce the topic of Laplace transform. So thank you very much.