Integral Transforms and Their Applications

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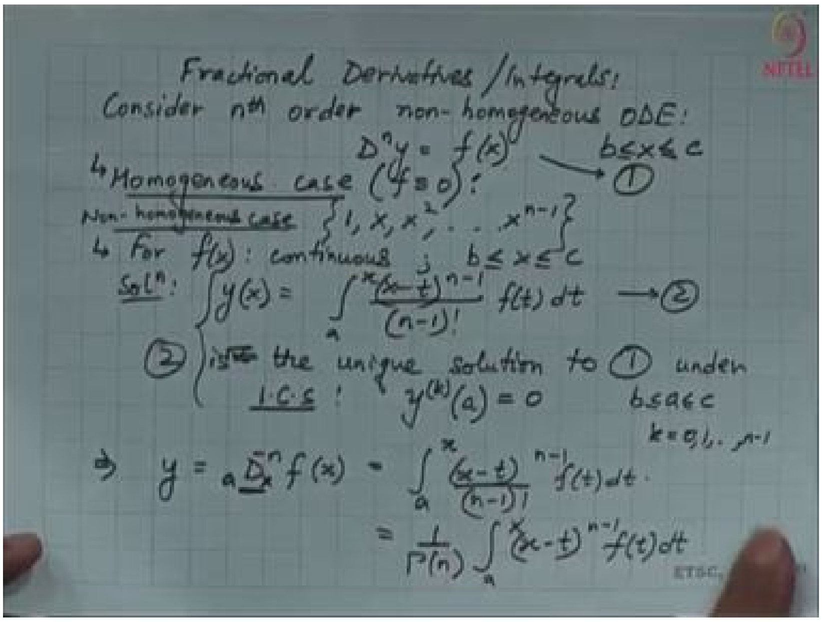
Lecture –56

Introduction to Fractional Calculus Part-02

So, what I am going to introduce is the general discussion on fractional derivatives and integrals. So, namely I am going to introduce fractional integrals; so, fractional derivatives and integrals. So, what I start with is the following, let us consider the nth order non homogeneous ODE, given by given by the following. So, let me write this down and I am given that the domain of this ODE this is in 1D,



$$D^n(y) = f(x) \quad ..(1) \qquad b \le x \le c$$

Now, I see that this particular ODE can be solved by looking at the two solutions: one for the homogeneous case and one for the non-homogeneous case. I know that for the homogeneous case; for the homogeneous case let me further, I need to give you the boundary condition.

So, for but let me just immediately state the result for the homogeneous case; for the homogeneous case that is when f is identically 0, I have that the solution the solution is in this pace off the basis functions.So, this is since this is an nth order ODE my solution is a linear combination of all these factors, I see that my homogeneous case is for f is identically equal to 0 is a linear combination of all these expressions, all these factors.

So, what I have is the following. So, then let us see for f(x) now I am going to start looking at the non-homogeneous case now.let us say For the non-homogeneous case f(x) continuous.

If I directly invert equation (1) I get back my solution y in terms of f. and when I do that my solution y has the following integral:

Solution:

$$y(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt \qquad ....(2)$$

I want to highlight that 2 is the unique solution to my equation 1 under the initial conditions given by the kth derivative of y evaluated at let us say any point a.

$$y^{(k)}(a) = 0 \quad b \leq a \leq c; \quad k = 0, 1, 2.., n-1$$

which means, so what I have shown here is that my solution to the non-homogeneous case is given by this integral. Now, if I were to rewrite this (1) in the inversion, suppose if I say that my D_n is an operator whose inverse exists.

So, which means that if I were to write; if I were to write my y to be the inversion of this operator D; inversion of this operator D of evaluated at f(x) this inversion is going to be nothing, but the integral that is defined by 2 here. So, the integral is :

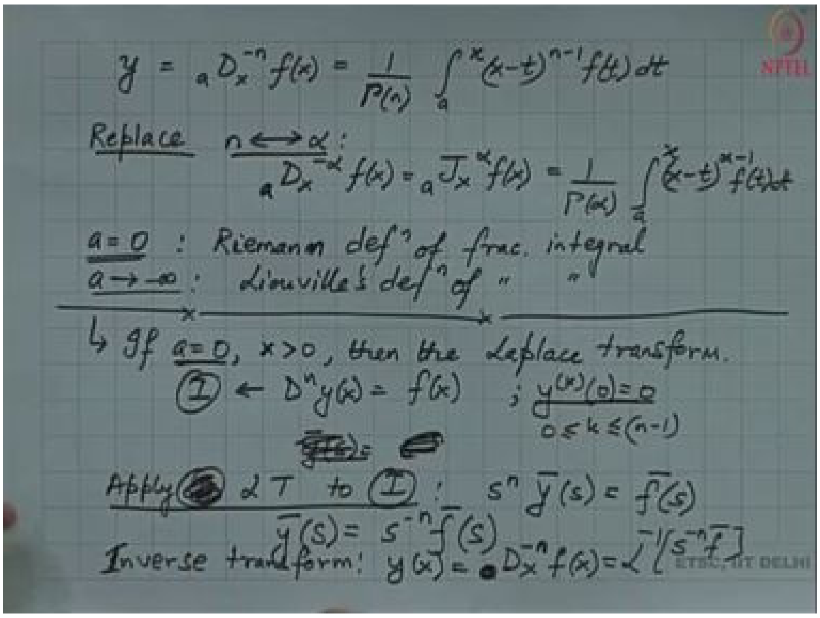
$$y = aD_x^{-n} f(x) = \int_a^x \frac{(x-t)}{(n-1)!}^{n-1} f(t)dt$$

So, that is my I see that there is a relation between the inverse of the operator with this integral ok. So, namely let me just define the inverse of this operator now.

$$= \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt$$

So, let me write down this expression again. So, what I have is the following.

$$y = {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt$$



I see that if I were to replace my n with a fractional α I am going to get the relation for the inversion operator for the fractional derivative. So, let me do that let me replace. So, then I get the formula for the fractional derivative with respect with order α.

$${}_aD_x^{-\alpha} f(x) = {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(x)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

So, then there are some specific definition; let us say that a is 0. So, if a=0 then that definition is the Riemann;the Riemann definition of fractional integral and let us say for a going to $-\infty$, that is my Liouville's definition of fractional integral.

So, let us see, let us try to evaluate the Laplace transform of the following ODE, I am given the following initial condition that is$0 \leqslant k \leqslant (n-1)$

$$D^n y(x) = f(x) \qquad ....(\mathrm{I})$$

Apply Laplace transform to the equation (I)

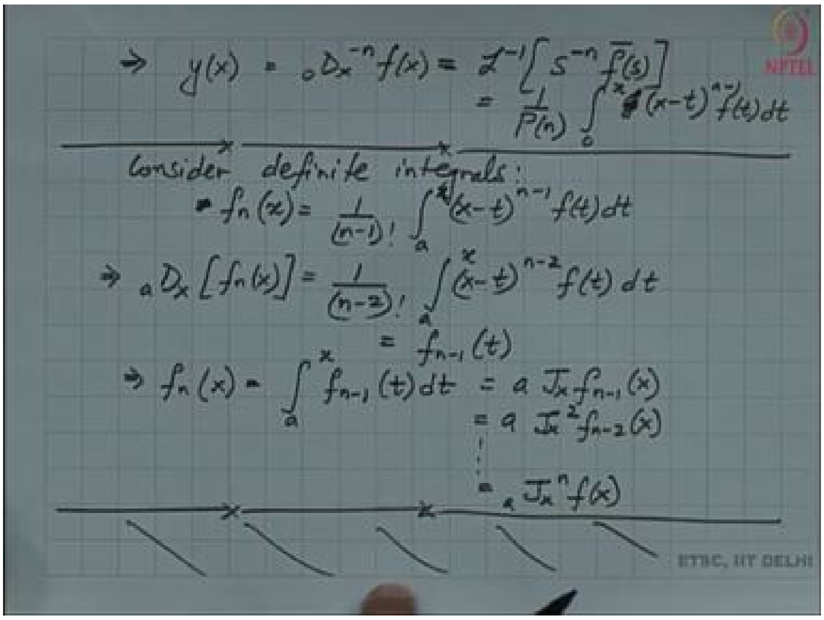
$$s^n \bar{y}(s) = \overline{f}(s)$$

$$\bar{y}(s) = s^{-n} \bar{f}(s)$$

Now the Inverse Transform,

$$y(x) = D_x^{-n} f(x) = \mathcal{L}^{-1}[s^{-n}\bar{f}]$$

$$y(x) = {}_aD_x^{-n} f(x) = \mathcal{L}^{-1}\left[s^{-n}\bar{f}(s)\right]$$



Consider Definite Integrals:

$$f_n(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt$$

$$\Rightarrow {}_aD_x[f_n(x)] = \frac{1}{(n-2)!}\int_a^x (x-t)^{n-2} f(t)dt = f_{n-1}(t)$$

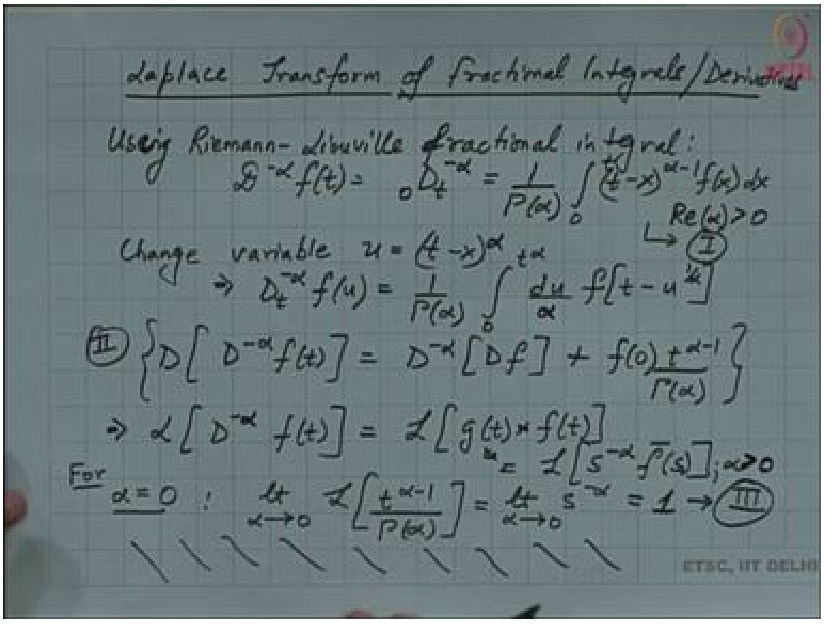
$$\Rightarrow f_n(x) = \int_a^x f_{n-1}(t)dt = aJ_x f_{n-1}(x)$$

$$= aJ_x{}^2 f_{n-2}(x)$$

if I were to continue this is nothing, but the nth the application of the nth ,

$$= {}_aJ_x^n f(x)$$

So, J x is my integral operator. So then, so that gives me a relation a recurrence relation between the derivatives in each of these orders. So, that gives me a recurrence relation on these integral operators here.



Laplace Transform of Fractional Integral/Derivatives:

let me consider the Laplace transform of integrals here. So, Laplace transform Laplace transform of fractional integrals, the Laplace transform of fractional integrals or derivatives to see what relation that we have. So,I am going to find let me just use, let me just use my Riemann-Liouville definition. So, using Riemann-Liouville definition of the fractional integral fractional integral to come to my expression here that the fractional derivative of f(t).

$$D^{-\alpha}f(t) = {}_0D_t^{-\alpha} = \frac{1}{\Gamma(x)}\int_0^t (t-x)^{(\alpha-1)}f(x)dx...(I) \qquad ; Re(\alpha) > 0$$

So, that is that is the definition of my fractional integral, let me just say that my real part of this fraction is positive. So, we see that there is almost no restriction on this α, the α can be fraction the α can even be complex using this definition as long as the real part is positive here .

So, let me just change the variable, $u = (t-x)^{\alpha}$

$\Rightarrow D_t^{-\alpha}f(u) = \frac{1}{\Gamma(\alpha)}\int_0^{t\alpha}\frac{du}{\alpha}f\left[t-u^{1/\alpha}\right]$

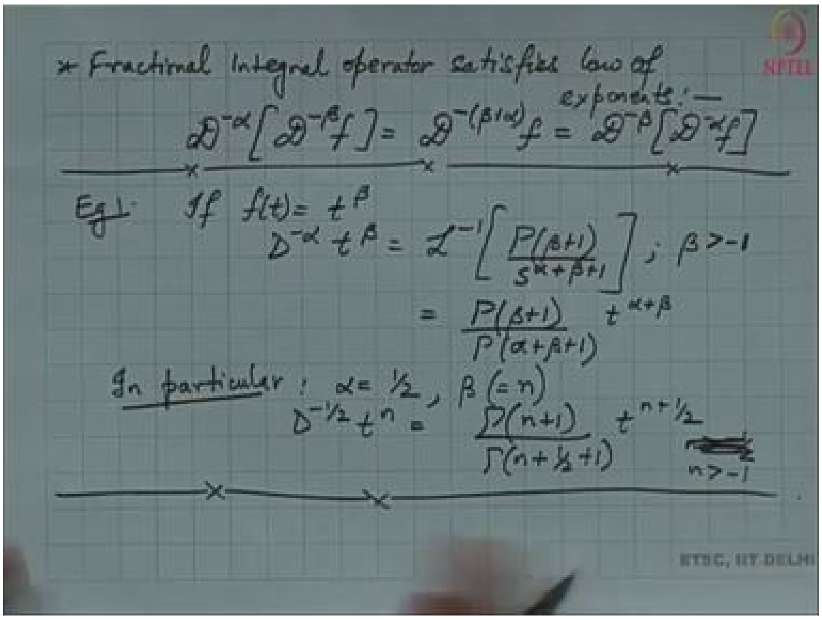
$D\left[D^{-\alpha}f(t)\right] = D^{-\alpha}[Df] + f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ ..(II)

$\Rightarrow \mathcal{L}\left[D^{-\alpha}f(t)\right] = \mathcal{L}[g(t) * f(t)]$

$= \mathcal{L}\left[s^{-\alpha}\bar{f}(s)\right] \qquad ; \alpha > 0$

For $\alpha = 0$,

$\lim_{\alpha \to 0} \mathcal{L}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] = \lim_{\alpha \to 0} s^{-\alpha} = 1 \quad ...(\mathrm{III})$



Fractional Integral operator satisfies Law of exponents,

$D^{-\alpha}\left[D^{-\beta} f\right] = D^{-(\beta+\alpha)} f = D^{-\beta}\left[D^{-\alpha} f\right]$

Example 1: If $f(t) = t^\beta$

Solution:

$$D^{-\alpha} t^\beta = \mathcal{L}^{-1}\left[\frac{\Gamma(\beta+1)}{s^{k+\beta+1}}\right] \quad ; \beta > -1$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$$

In Particular,$\alpha = 1/2, \beta(= n)$

$$D^{-1/2} t^n = \frac{\Gamma(n+1)}{\Gamma(n+1/2+1)} t^{n+\frac{1}{2}} \qquad ; n > -1$$