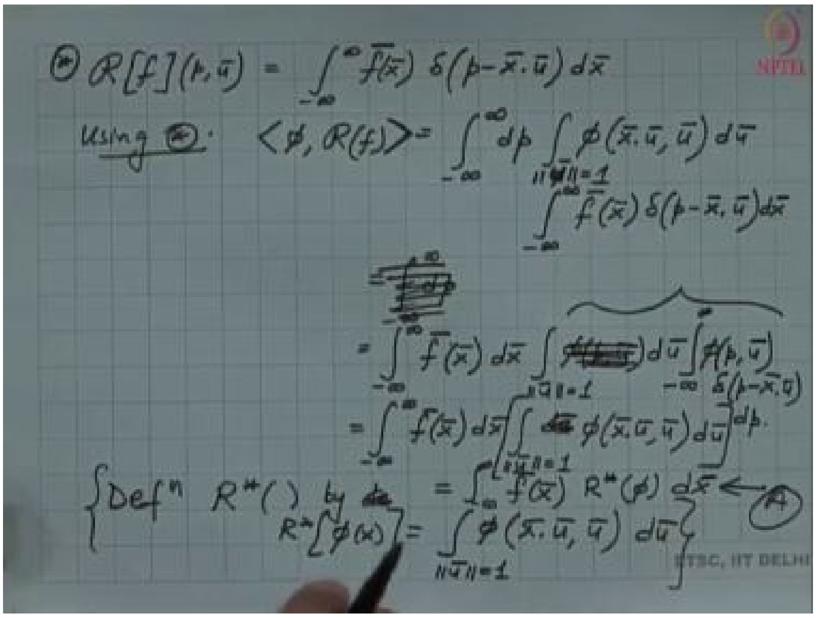
Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture –54

Inverse Radon Transform, Applications to Radon Transform Part 3



$$R[f](p, \bar{u}) = \int_{-\infty}^{\infty} \overline{f(x)} \delta(p - \bar{x} \cdot \bar{u}) d\bar{x}$$

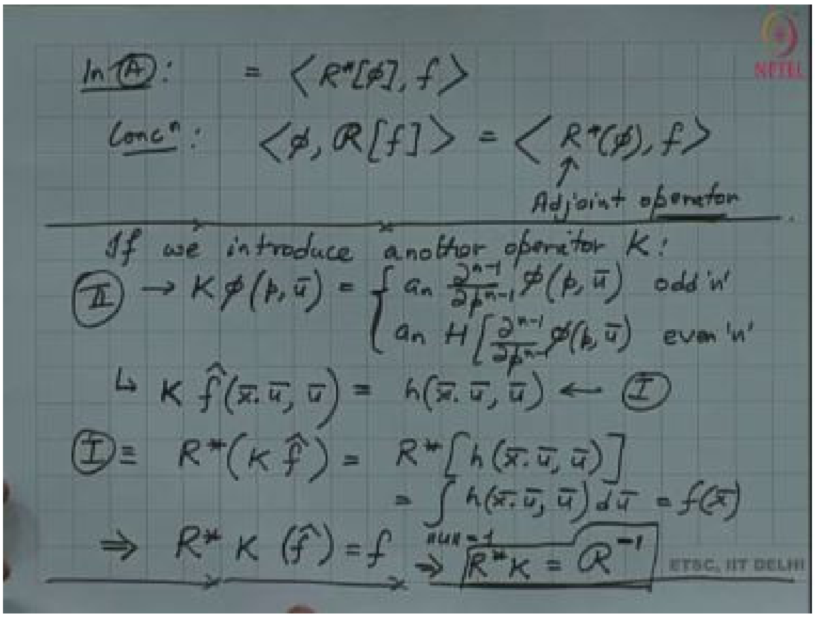
$$\langle \phi, \mathbb{R}(f) \rangle =$$

$$\int_{-\infty}^{\infty} dp \int_{\|\Phi\|'=1} \phi(\bar{x} \cdot \bar{u}, \bar{u}) d\bar{u}$$

$$\int_{-\infty}^{\infty} f(x) \delta(p - \bar{x}, \bar{u}) dx$$

$$= \int_{-\infty}^{\infty} F(x) dx \int_{||u=1||} d\bar{u} \int_{-\infty}^{\infty} \phi(p, \bar{u}) \delta(p - \bar{x}, \bar{u})$$

$$R^*[\phi(x)] = \int_{||\bar{u}=1||} \phi(x \cdot \bar{u}, \bar{u}) d\bar{u}$$



$$= \langle R^*[\phi], f\rangle$$

$$\langle \phi, Q[f]\rangle = \langle R^*(\phi), f\rangle$$

If we introduce another operator K:

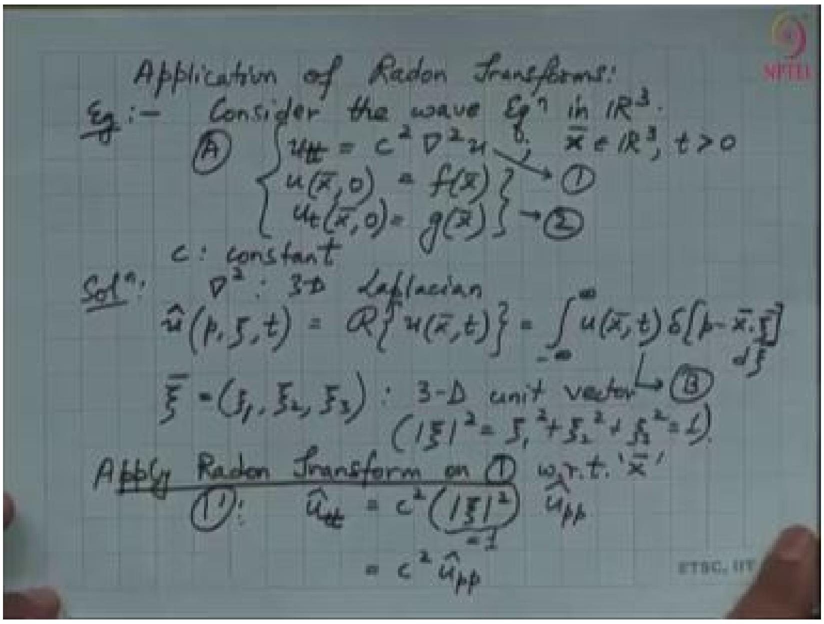
$$K\phi(p,\bar{u}) = \begin{cases} a_n \frac{\partial^{n-1}}{\partial p^{n-1}}\phi(p,\bar{u}) & \text{odd } n' \\ a_n H\left[\frac{\partial^{n-1}}{\partial p^{n-1}}\phi(p,\bar{u})\right. & \text{even n }' \end{cases}$$

$$k\hat{f}(\bar{x}\cdot\bar{u},\bar{u}) = h(\bar{x}\cdot\bar{u},\bar{u}) \leftarrow 1$$

$$R^*(k\hat{f}) = R^*[h(\bar{x}\cdot\bar{u},\bar{u})]$$

$$= \int h(\bar{x}\cdot\bar{u},\bar{u})d\bar{u} = f(\bar{x})$$

$$R^*K(\hat{f}) = f \rightarrow R^*K = R^{-1}$$



Application of Radon Transform: Example: Consider the wave equation in IR3:

$$
\begin{aligned}
& u_{tt} = c^2\nabla^2 u \quad \delta_i \quad \bar{x} \in I\mathbb{R}^3, t > 0 \\
& u_t(\bar{x},0) = f(\bar{x}) \\
& u_t(x,0) = g(\bar{x})\}
\end{aligned}
$$

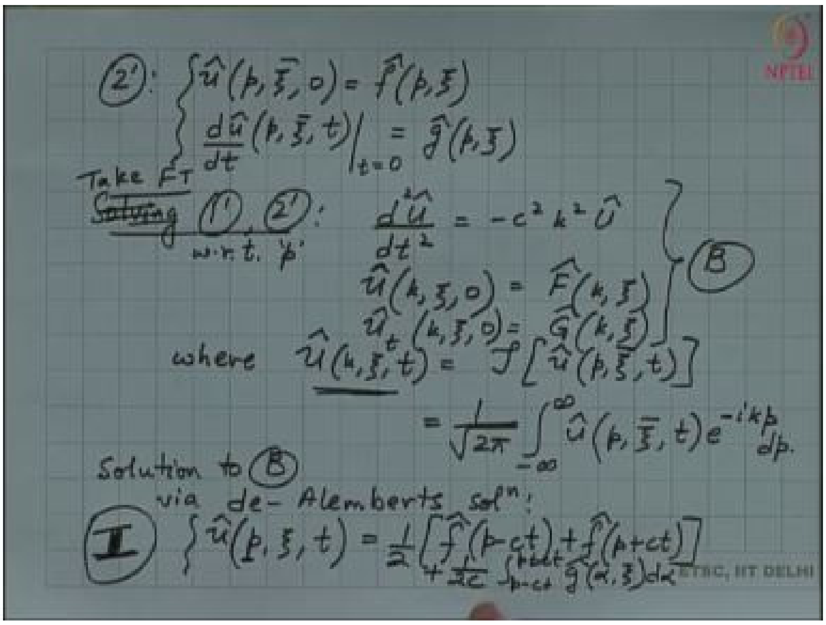
Solution:

$$\hat{u}(p,\xi,t)^{3t}$$

$$R\{u(\bar{x},t)\} = \int_{-\infty}^{\infty} u(\bar{x},t)\delta[p-\bar{x}\cdot\bar{\xi}]d\xi$$

Apply Radon Transform:

$$
\begin{aligned}
\hat{u}_{tt} &= c^2 \left(|\xi|^2\right) \hat{u}_{pp} \\
&= c^2 \hat{u}_{pp}
\end{aligned}
$$



2.

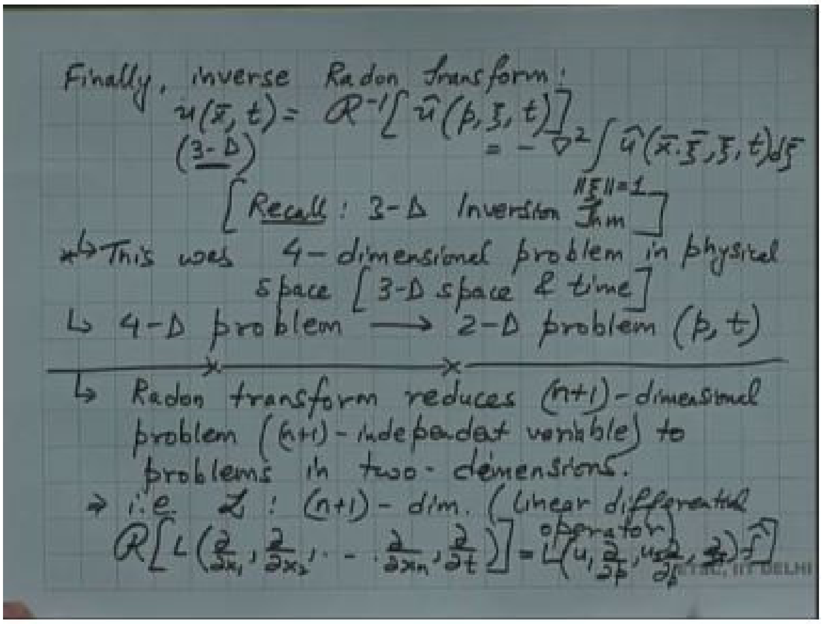
$$\hat{u}(p,\bar{\xi},0)=\hat{f}(p,\xi)$$
$$\frac{d\hat{u}}{dt}(p,\bar{\xi},t)\Big|_{t=0}=\hat{g}(p,\bar{\xi})$$

$$\left.\begin{array}{l}\frac{d^2\hat{u}}{dt^2}=-c^2k^2\hat{U}\\ \hat{u}(k,\xi,0)=\hat{F}\left(\ k,\ \ \xi\ \right)\end{array}\right\}B$$

$$u_t(k,\xi,0)=G(k,\xi)$$

$$\text{where, }\hat{u}(k,\xi,t)=\rho[\hat{u}(p,\xi,t)]$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k,\bar{\xi},t)e^{-ik\beta}$$

$$\Big\{\hat{u}(p,\xi,t)=\frac{1}{2}[f(p-ct)+\hat{f}(p+ct)]+\frac{1}{2c}\int_{p-ct}^{p+ct}\bar{g}(\alpha\bar{\xi})d\alpha$$

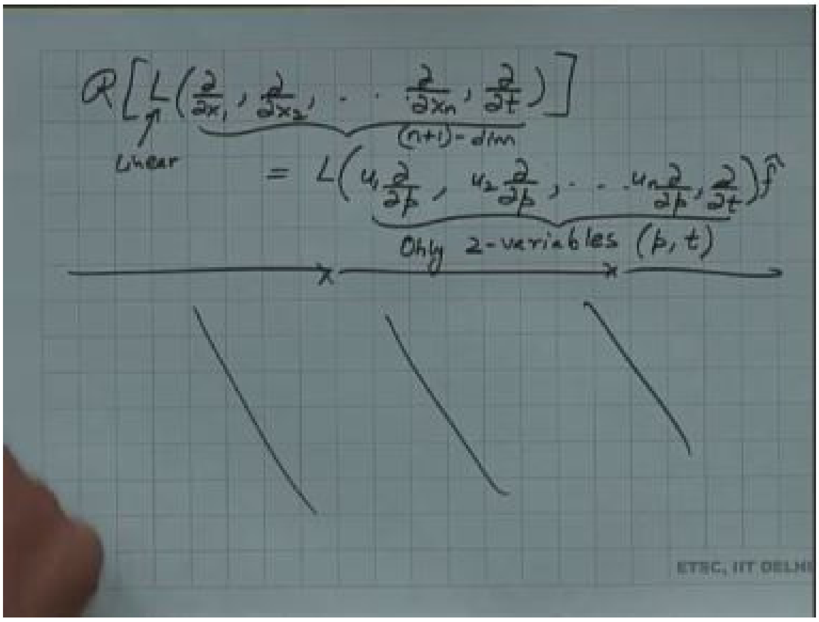


Finally Inverse Radon Transform:

$$u(\bar{x},t)=R^{-1}[\hat{u}(p,\xi,t)]==-\nabla^2\int_{||\xi=1||}\hat{u}(\bar{x}\cdot\bar{\xi},\xi,t)d\bar{\xi}$$

I have to finally, take an inverse Radon transform to come to the solution. So, after taking the inverse Radon transform, I get back my solution that is the solution in the physical plane. Although this method seems slightly involved in finding the solution. But, the most important part is that this was this particular problem this was a 4 dimensional problem, this was a 4 dimensional problem, this was a 4 dimensional problem in physical space. This was a 4 dimensional problem in physical space that is 3 D space and time. So, what have I done is I have reduced this 4 D problem into a 2 D problem into a 2 D problem 2 D problem in the Radon transform plane ,It reduces the dimensionality of the problem from a very high dimension to a 2 dimensional problem, but the price that we have to pay is that the calculations in finding the inverse as well as the Radon transforms is slightly involved. So, let me just highlight it highlight and write what I just mentioned. So, whatever I have shown in this example I can generalize this result. So, the generalized result is as follows:

$$Q\left[L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_n}\frac{\partial}{\partial t}\right)\right] == L\left(u_1\frac{\partial}{\partial p}, u_2\frac{\partial}{\partial p}, \frac{\partial}{\partial t}\right)\hat{f}$$



$$Q\left[L_1\left(\frac{2}{3x_1}, \frac{2}{2x_2}, \cdots \frac{2}{2x_1}, \frac{2}{2t}\right)\right]$$
$$= L\left(u_1\frac{\partial}{\partial p}, u_2\frac{\partial}{\partial p}, \cdots u_n\frac{\partial}{\partial p}, \frac{\partial}{\partial t}\right)\hat{f}$$

So, that concludes that concludes my discussion on Radon transform. Students are great highly encouraged to look for more problems involving these transform functions. But, in the next lecture I am going to come to a new topic that is on fractional calculus. This is interesting in the sense that, we all know how to evaluate the derivatives of a function. We all well, we I am assuming students have a background in calculus. So, people having a background in calculus, they all know how to evaluate derivatives or they all know how to evaluate integrals simple integrals. But what about the derivatives of fractional order or what about the integrals of fractional order right? So, in the next lecture, I am going to highlight those expressions as to how to evaluate those fractional derivatives and fractional integrals. And, followed by that I am going to show you some application problems which involves those fractional derivatives and fractional integrals. So, thank you for listening. Thank you very much.