Integral Transforms and Their Applications

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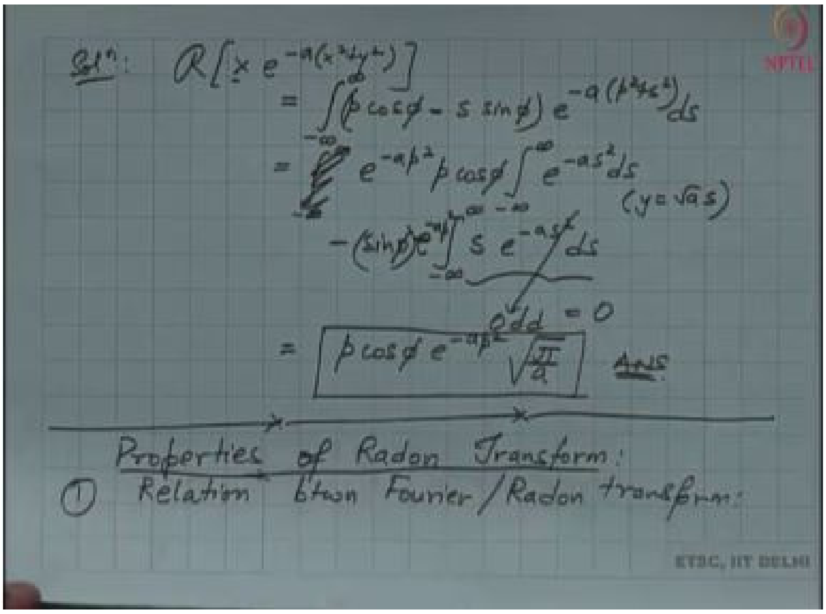
Lecture –50

Introduction to Radon Transform Part 2



Solution:

$$R\left[xe^{-a\left(x^2+y^2\right)}\right]$$

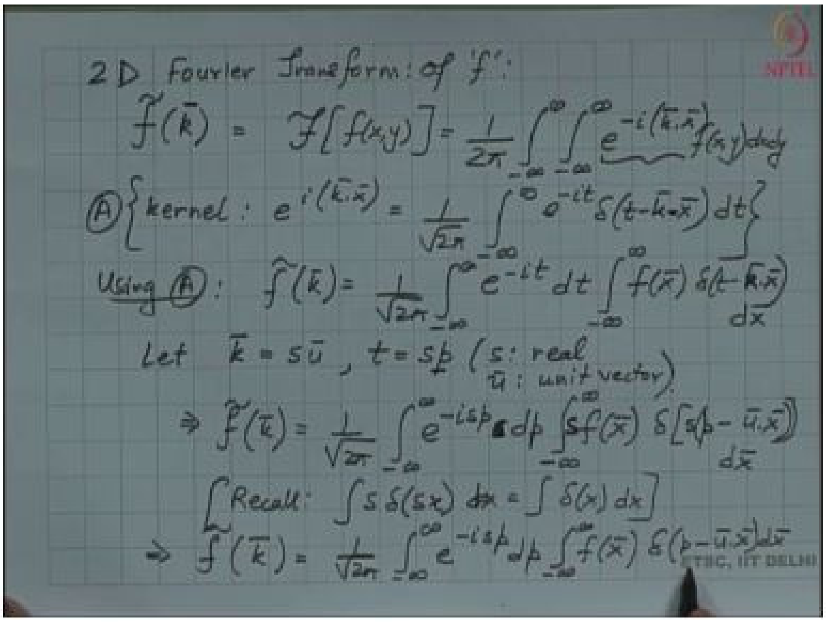
$$= \int_{-\infty}^{\infty} \left(p\cos\phi - s\sin\phi\right) e^{-a\left(p^2+s^2\right)} ds$$

$$e^{-ap^2} p\cos\phi \int_{-\infty}^{\infty} e^{-as^2} ds$$

$$-(\sin\phi) e^{-ap^2} \int_{-\infty}^{\infty} s e^{-as^2} ds$$

$$p\cos\phi e^{-ap^2} \sqrt{\frac{\pi}{a}}$$

Properties of Random Transform: let us look at the most important property. The most important property of Radon transform is its relation with Fourier transform. relation between Fourier and Radon transform. let us look at the relation between Fourier and Radon transform here. So, let us see what happens.



$$f(\bar{k}) = [f(x,y)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\infty}^{\infty} e^{-i(\bar{k}\cdot\bar{x})} f(x,y)dxdy$$

$$e^{i(\bar{k}\cdot\bar{x})} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} e^{-it}\delta(t-\bar{k}-\bar{x})dt$$

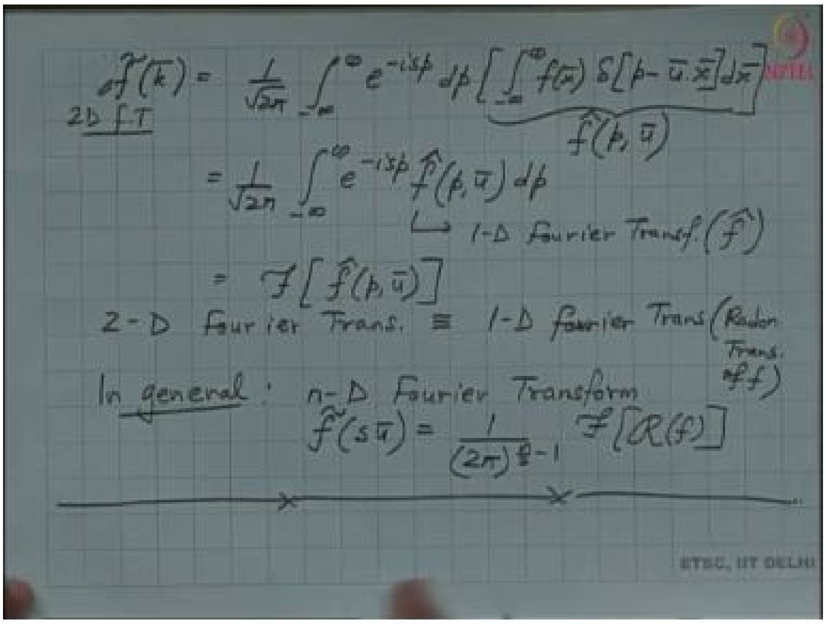
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it}dt\int_{-\infty}^{\infty} f(\bar{x})\delta(t-\bar{k},\bar{x})dx$$

$$\text{let } \bar{k} = s\bar{u}, t = sp$$

s:real, u:unit vector

$$f(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isp}dp\int_{-\infty}^{\infty} sf(x)\delta[s(p-\bar{u},\bar{x})]dx$$

$$f(\bar{k}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isp}dp\int_{-\infty}^{\infty} f(x)\delta\left(p-\bar{u}\cdot\bar{x}\right)dx$$

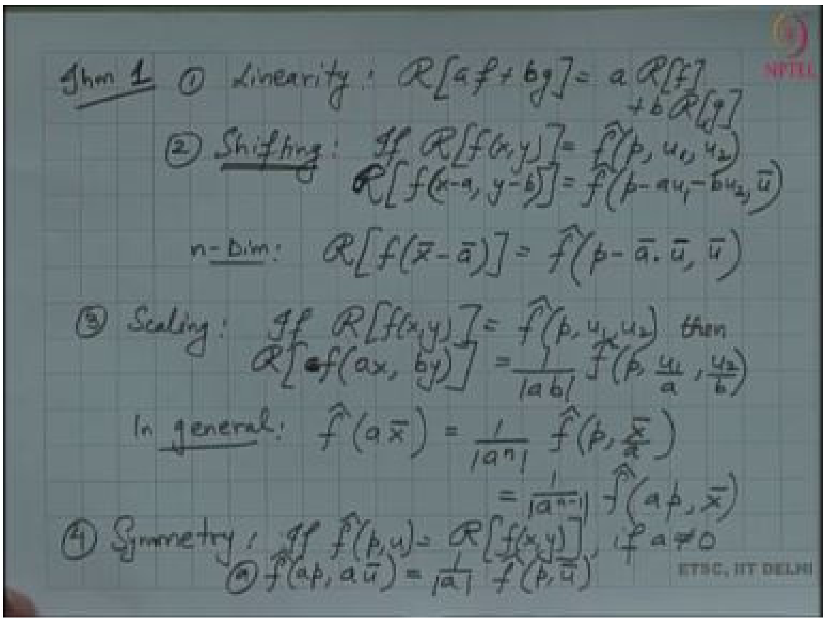


$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} f(p, \bar{u}) d\beta$$

$$f[\tilde{f}(p, \bar{u})]$$

In general: n-D fourier transform

$$\tilde{f}(s\bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}-1}} f[R(f)]$$



Theorem: 1.Linearity:

$$R[af + bg] = aR[f] + bR[g]$$

2. Shifting:

$$\text{If } R[f(x, y)] = \tilde{f}\,(p, u_1, u_2)$$

$$R[f(x-a, y-b)] = f\left(p - au_1 - bu_2, \bar{u}\right)$$

n-dim:

$$R[f(\bar{x} - \bar{a})] = \widehat{f}(p - \bar{a} \cdot \bar{u}, \bar{u})$$

3. Scaling:

$$\text{If } \mathbb{R}[f(x, y)] = f(p, u_1, u_2)$$

$$R[f(ax, by)] = \frac{1}{|ab|} f\left(p, \frac{u_1}{a}, \frac{u_2}{b}\right)$$

In general:

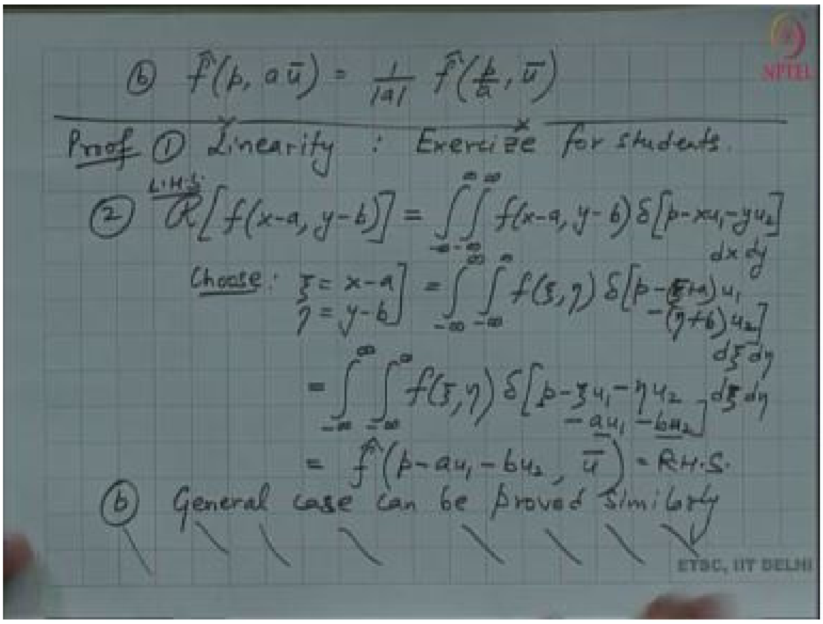
$$\hat{f}(a\bar{x}) = \frac{1}{|a^n|} \hat{f}\left(p, \frac{\bar{x}}{a}\right)$$

$$= \frac{1}{(a^{n-1})} \hat{f}(ap, \bar{x})$$

4. Symmetry:

$$\text{If } \hat{f}(p, u) = Rf[x, y], a \neq 0$$

$$\text{a.} \hat{f}(ap, a\bar{u}) = \frac{1}{|a|} f\{(p, \bar{u})\}$$



$$\text{b.} \hat{f}(p, a\bar{u}) = \frac{1}{|a|} \hat{f}\left(\frac{p}{a}, \bar{u}\right)$$

Proof: 1. Exercise for Students

2. Proof:

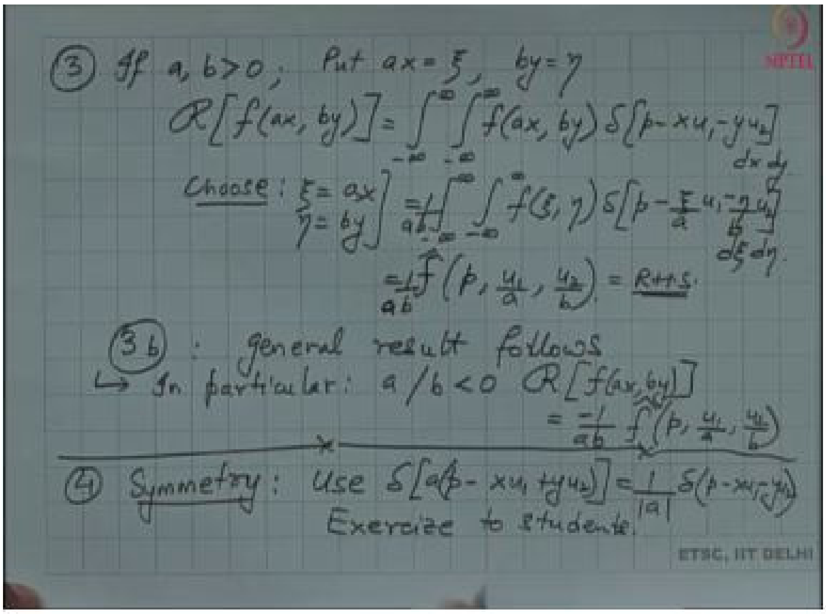
$$R[f(x-a, y-b)] =$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-a, y-b) \delta\left[p - xu_1 - yu_2\right] dx dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta\left[p-\xi u_1-\eta u_2-au_1-bu_2\right]d\xi d\eta$$

$$= \hat{f}\left(p-au_1-bu_2,\bar{u}\right)$$

General case can be proved similarly,



$$3.\ \text{If } a,b>0; \quad \text{Put } ax=\xi, \quad by=\eta$$

$$R[f(ax,by)] =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ax,by)\delta\left[p-xu_1-yu_2\right]dxdy$$

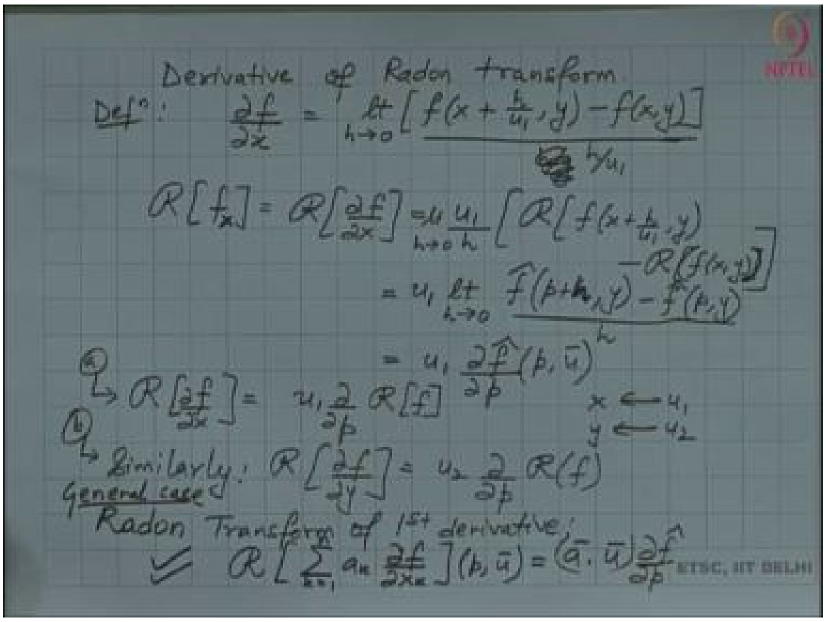
$$= \frac{1}{ab}\hat{f}\left(p,\frac{u_1}{a},\frac{u_2}{k}\right)$$

$$a/b<0 \quad R[f(ax,by)]$$
$$= \tfrac{-1}{ab}f\left(b,\tfrac{u_1}{a},\tfrac{u_2}{b}\right)$$

4. Symmetry

$$\text{Use } \delta\left[a\left(p-xu_1+yu_2\right)\right] = \frac{1}{|a|}\delta\left(p-xu_1-yu_2\right)$$

Exercise to students



Derivative of Random transform:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \left[f\left(x + \frac{h}{u_1}, y\right) - f(x, y) \right]$$

$$R\left[f_x\right] = R\left[\frac{\partial f}{\partial x}\right]$$

$$= lt_{h \to 0} \frac{u_1}{h} \left[\mathbb{R}\left[f\left(x + \frac{1}{u_1}, y\right) - R[f(x, y)]\right.\right.$$

$$= u_1 \,\mathrm{lt}_{h \to 0}\, \hat{f}\,(p + h, y) - f(p, y)$$

$$u_1, \frac{\partial \hat{f}}{\partial p}(p, \bar{u})$$

a.

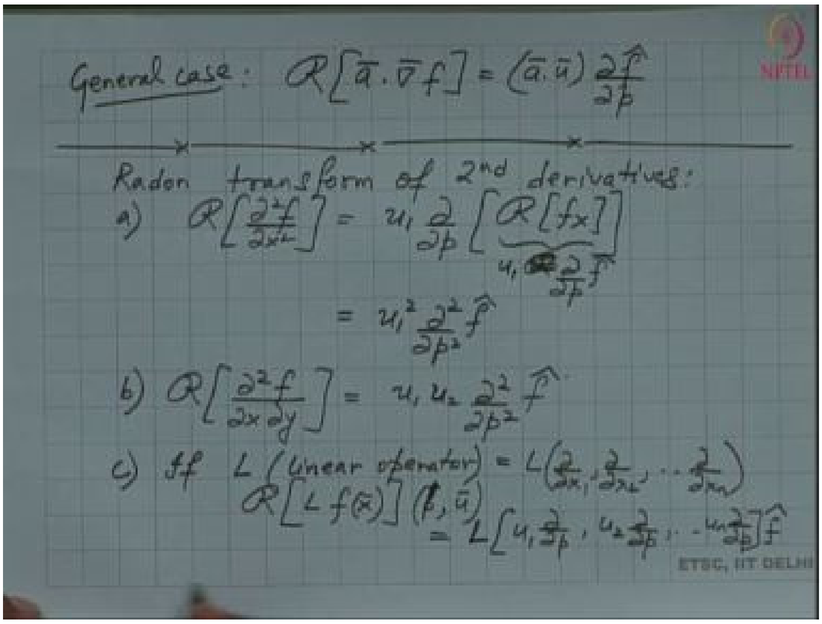
$$R\left[\frac{\partial f}{\partial x}\right] = u_1 \frac{\partial}{\partial p} R[f]$$

Similarly, b.

$$R\left[\frac{\partial f}{\partial y}\right] = u_2 \frac{\partial}{\partial p} \mathbb{R}(f)$$

Generally, Random Transform 1st Derivative:

$$R\left[\sum_{k=1}^{m} a_k \frac{\partial f}{\partial x_k}\right](p, \bar{u}) = (\bar{a}, \bar{u}) \frac{\partial \hat{f}}{\partial p}$$



General Case:

$$R[\bar{a} \cdot \bar{\nabla} f] = (\bar{a} \cdot \bar{u}) \frac{\partial \widehat{f}}{\partial p}$$

Random transform of 2nd derivative:

$$\text{a. } R\left[\frac{\partial^2 f}{\partial x^2}\right] = u_1 \frac{\partial}{\partial p} \underbrace{[R[fx]]}$$

$$= u_1^2 \frac{\partial^2}{\partial p^2} \hat{f}$$

$$\text{b. } R\left[\frac{\partial^2 f}{\partial x \partial y}\right] = u_1 u_2 \frac{\partial^2}{\partial p^2} \hat{f}$$

If L(Linear operator):

$$L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots \frac{\partial}{\partial x_n}\right),$$

$$R[Lf(x)](p, \bar{u}) = L\left(\frac{\partial}{u_1 \partial p}, \frac{\partial}{u_2 \partial p}, \cdots \frac{\partial}{u_n \partial p}\right) \hat{f}$$