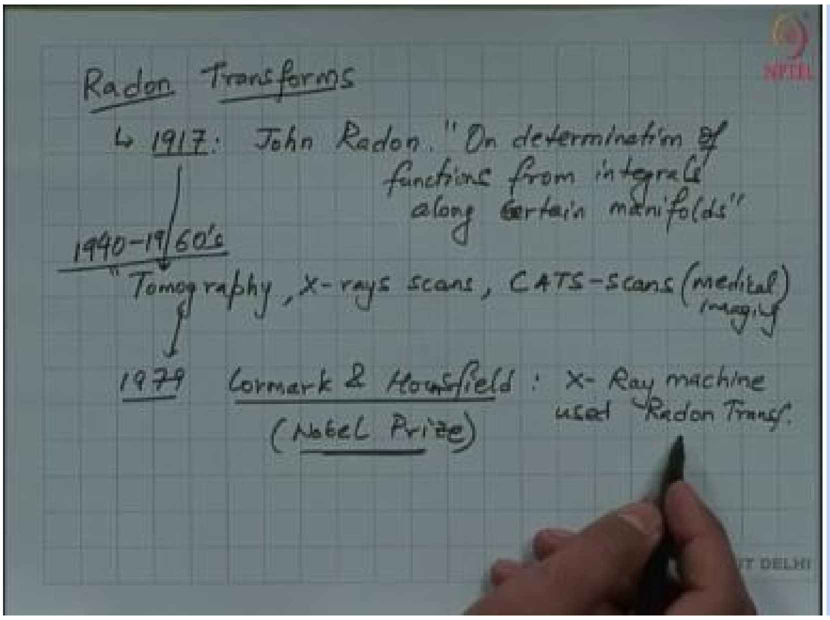
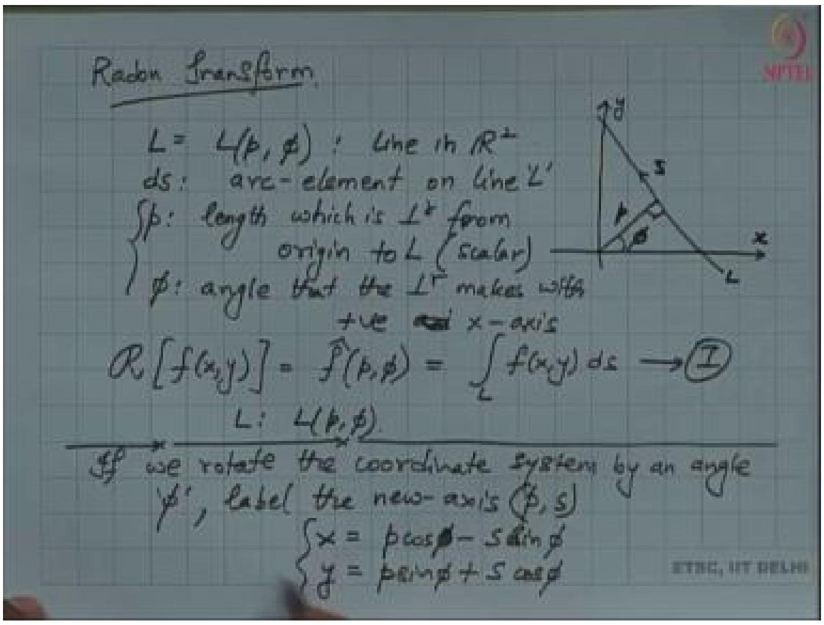
Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture –49

Introduction to Radon Transform Part 1

Good morning everyone. So, today in this lecture I am going to introduce another new transform namely the Radon Transform. So, far I have we have seen that the and we have developed and describe different types of transforms either on the real axis or on the complex plane or on the imaginary axis or mainly on the well known Cartesian coordinates or polar coordinates or cylindrical coordinates right. So, what happens if we were to describe a transform in a arbitrary manifold or a plane that we do not know where is the orientation or we have a random or well known or a specified you know orientation of that plane. So, the question is can we go ahead and describe a transform for a arbitrary manifold or an arbitrary hyper plane. So, to describe such a transform I am going to start the basic background.



So, to begin with let me just highlight some of the major you know the major achievements here. So, Radon transforms before I start defining Radon transforms let me just you know come back and describe a bit more about these transfer because these are a little special. In the sense that the application of Radon transforms preceded the development of the transform, it was because that in 19 early 1900's an X ray machine was developed and it was not known how to take images and how to use those images for an arbitrary hyper plane. So, it was only after the development of these Radon transforms that those in that information of data from those images at those at those hyper planes were suitably developed. So, let me just mention some of the major landmarks in the development of Radon transform. So, the first landmark in the development of Radon transform was in 1917 with where it where the first paper by John Radon was published on determinants well that I for students were curious I am going to write down even the topic of the paper; on determination of functions from integrals along certain manifolds ok. So, it was this phenomenal paper through which John Radon discovered the fundamental Radon transforms. Now it was not known what is the use of this Radon transform until another well from 1917 there was lots of you know little applications including applications in tomography. So, in all these areas this particular areas in tomography in X ray scans, the equipments were being developed said the X ray scanning machines, that computational tomography machines and the CT scan machines the CATS scans, but it was not known how to use and post process the images that were developed by these machine. So, by the way the CATS scans have wide ranging application in the area of medical imaging. So, although the machines were developed it was not known how to process those images developed by this machines. So, it was only, so, it was all from 1940's from 1940's to 1960's that these machines were being developed, but there was no proper usage of the data that was generated by this machines. It was only until 1979 that you know the two scientist by the name of Cormack and Hounsfield. So, these two scientists Cormack and Hounsfield they were the first people who were able to develop an X ray machine that successfully used the concept of Radon transform that successfully used X ray machine that use the Radon transform to process the images that were developed by this machines. So, due to their work they received the Nobel Prize. So it was their phenomenal work and noticed that it was only after the successful usage of this Radon transform that these people were able to you know come up with a significant you know career defining you know machine through which they were able to process the X ray data. So, the importance of Radon transform cannot be underestimated; cannot be underestimated because of this following landmark development. So, let me just start with the definition of Radon transform.



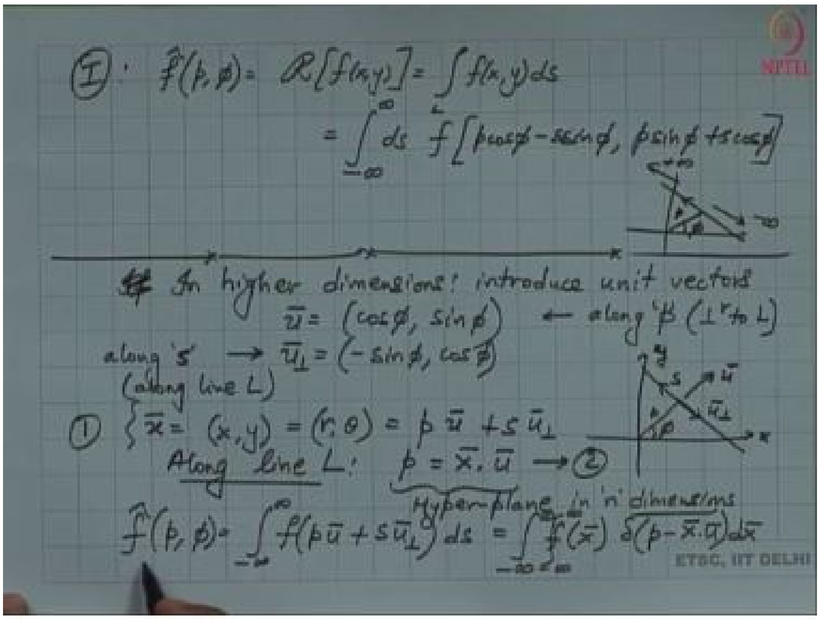
$$R[f(x,y)] = \hat{f}(p,\phi) = \int_L f(x,y)ds$$

So, the so, we can easily see that this sort of a definition can be extended to any two a line in any dimension namely a hyper plane. So, specially suppose if I have a data or an image which is described in n dimensions and we want to check the an extract information in any particular plane I can always project the data in that particular plane by taking the Radon transform. So, Radon transformers are specially useful if you were to project some you know data which is not meaningful to at to another plane, so, that the data can get meaningful. So, so hence the use usefulness of Radon transform.

$$L : \quad L(p, \phi)$$

If we rotate the coordinate system by an angle ϕ' , label the new-axis (p, s)

$$\begin{cases} x = p\cos\phi - s\sin\phi \\ y = p\sin\phi + s\cos\phi \end{cases}$$



1:

$$\begin{aligned} f(p,\phi) =& R[f(x,y)] = \int_L f(x,y)ds \\ &= \int_{-\infty}^{\infty} dsf[pcos\phi\text{-ssin }\phi, p\sin\phi + s\cos\phi] \end{aligned}$$

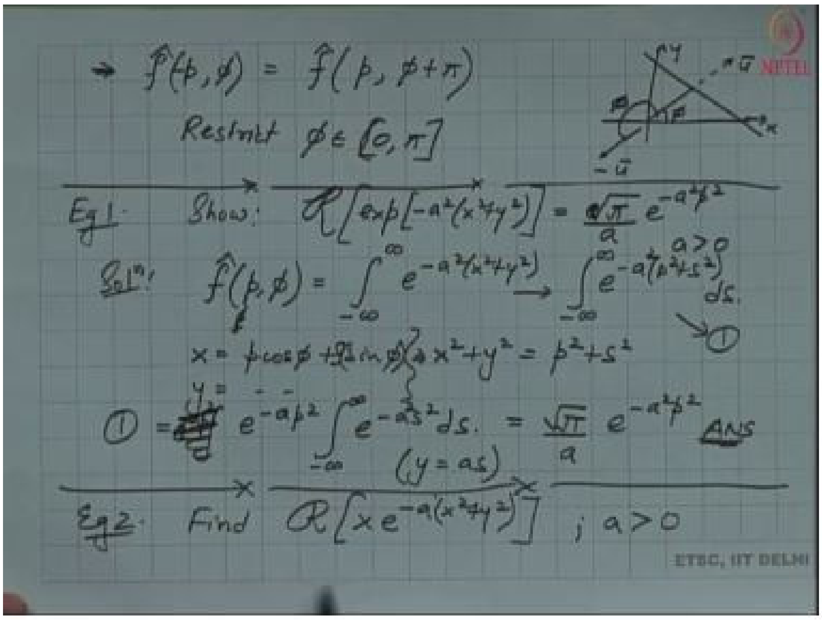
In higher dimensions: Introduce unit vectors:

$$\begin{aligned} \bar{u} &= (\cos\phi, \sin\phi) \\ \bar{u}_1 &= (-\sin\phi, \cos\phi) \end{aligned}$$

$$\bar{x} = (x, y) = (r, \theta) = p\bar{u} + s\bar{u}_1$$

$$\hat{f}(p,\phi) = \int_{-\infty}^{\infty} f\left(p\bar{u} + s\bar{u}_1\right) ds$$

$$\int_{-\infty} f(x) \cdot \delta(p - \bar{x} \cdot \bar{u}) dx$$



$$\hat{f}(p, \phi) = f(p, \phi + \pi)$$

$$\text{Restrict } \phi \in [0, \pi]$$

$$R\Bigg(\exp\left[-a^2\left(x^2+y^2\right)\right] = \frac{\sqrt{\pi}}{a}e^{-a^2p^2}$$

Solution:

$$f(p, \phi) = \int_{-\infty}^{\infty} e^{-a^2\left(x^2+y^2\right)} \rightarrow$$

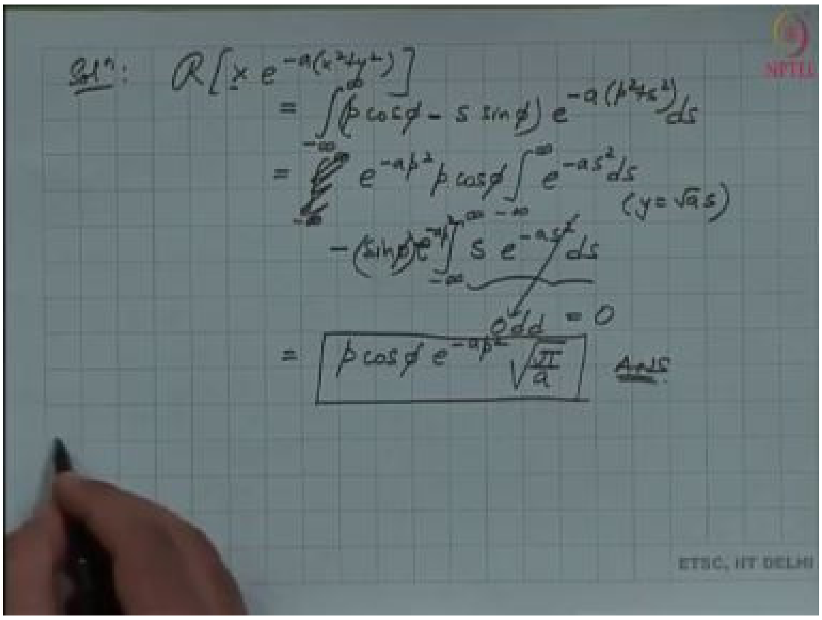
$$x = p\cos\phi + s\sin\phi$$

$$x^2 + y^2 = p^2 + s^2$$

$$e^{-ap^2}\int_{-\infty}^{\infty} e^{-a^2s^2}ds = \frac{\sqrt{\pi}}{a}e^{-a^2p^2}$$

Example 2:

$$\text{Find } R\left[xe^{-a\left(x^2+y^2\right)}\right]; a > 0$$



Solution:

$$R\left[xe^{-a\left(x^2+y^2\right)}\right]$$

$$= \int_{-\infty}^{\infty} (p\cos\phi - s\sin\phi)e^{-a\left(p^2+s^2\right)}ds$$

$$= e^{-ap^2} p\cos\phi \int_{-\infty}^{\infty} e^{-as^2}ds$$

$$= -(\sin\phi)e^{-ap^2} \int_{-\infty}^{\infty} se^{-as^2}ds$$

$$= p\cos\phi e^{-ap^2}\sqrt{\frac{\pi}{a}}$$