**Integral Transforms and Their Applications**

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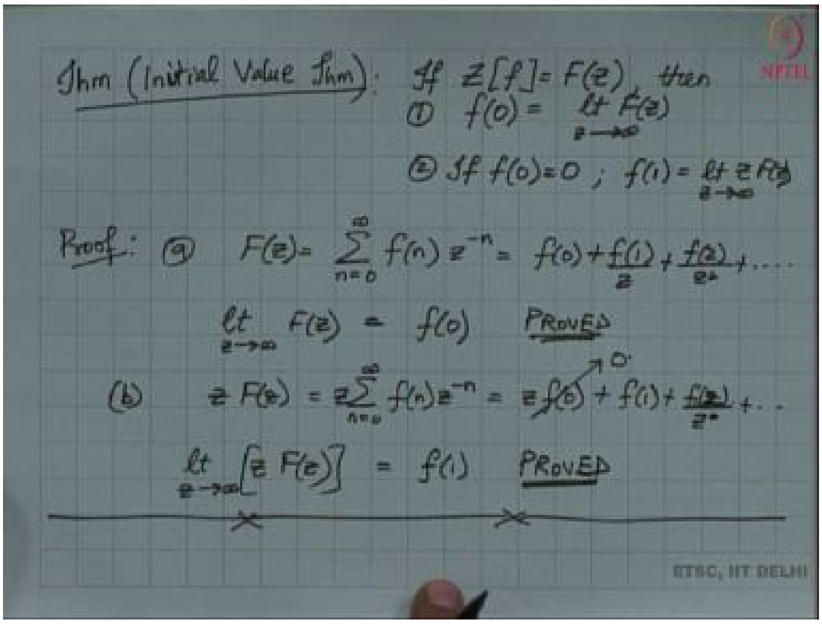
**Lecture –46**

## Inverse Z - transform, Applications of Z - Transform Part 1

Good morning everyone; so, in today's lecture I am going to continue my discussion on Z transform which I started in my previous lecture. So, namely I am going to in continue my discussion on certain useful results in the form of theorems and also, I am going to give you few more examples on how to apply these Z transforms followed by how to evaluate the inverse of the Z transform.




Initial Value Theorem:

If $z[f] = F(z)$ then
(1) $f(0) = lim_{z\to\infty} f(z)$
(2) If $f(0) = 0; f(1) = \lim_{z\to\infty} zF(z)$

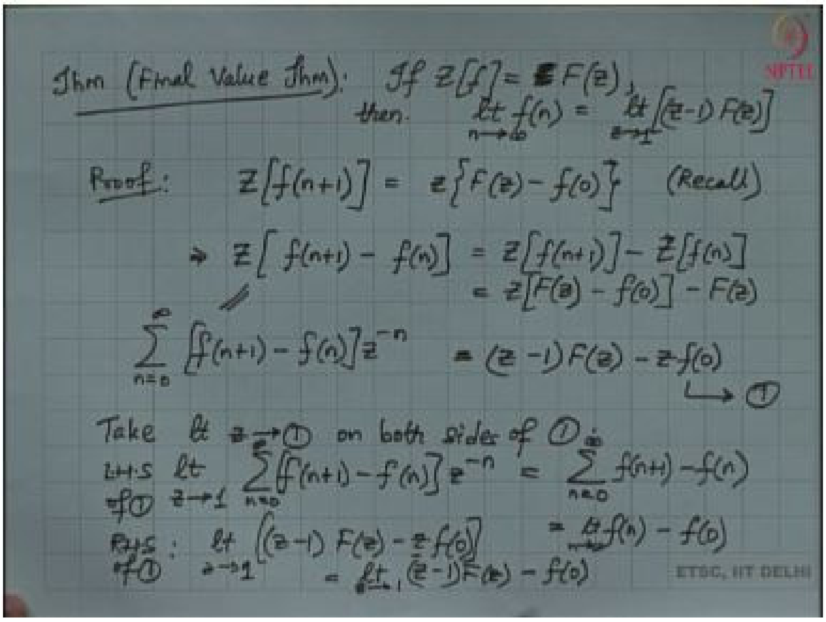
Proof:

$$\text{(a) } F(z) = \sum_{n=0}^{\infty} f(n)z^{-n} = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots$$

$$\lim_{z\to\infty} F(z) = f(0)$$

$$\text{(2) } zF(z) = z\sum_{n=0}^{\infty} f(n)z^{-n} = zf(0) + f(1) + \frac{f(z)}{z^2} + \cdots$$

$$\lim_{z\to\infty} [zF(z)] = f(1)$$




Final Value Theorem:

$$\text{If } z[f] = zF(z)$$
$$\text{then, } \lim_{n\to\infty} f(n) = \lim_{z\to 1}((z-1)F(z))$$

Proof:

$$z[f(n+1)] = z\{f(z) - f(0)\}$$
$$z[f(n+1) - f(n)] = z[f(n+1)] - z[f(n)]$$
$$= z[F(z) - f(0)] - F(z)$$

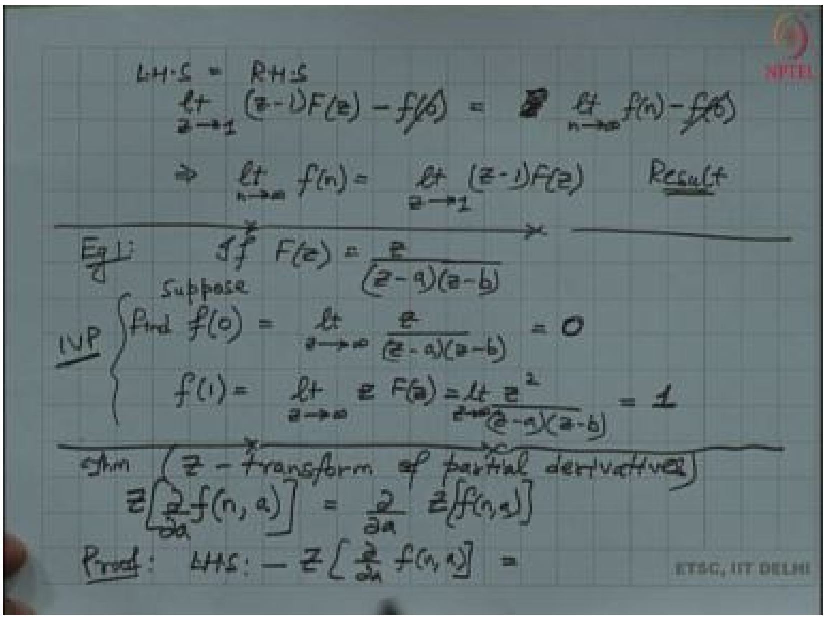
$$\sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n} \quad = (z-1)F(z) - zf(0) \to \ (1)$$

Take $\lim_{z\to 1}$ on both sides of equation 1, LHS

$$\lim_{z\to 1}\sum_{n=0}^{\infty}(f(n+1) - f(n)]z^{-n} = \sum_{n=0}^{\infty} f(n+1) - f(n)$$

RHS

$$\lim_{z\to 1}[(z-1)F(z) - zf(0)]$$
$$= \lim_{n\to\infty} f(n) - f(0)$$
$$= \lim_{z\to 1}\,(z-1)F(z) - f(0)$$




LHS=RHS

$$\lim_{z\to 1}(z-1)F(z)-f(0)=\lim_{n\to\infty}f(n)-f(0)$$

$$\lim_{n\to\infty}f(n)=\lim_{z\to 1}(z-1)F(z)$$

Example 1:

$$\text{If } F(z)=\frac{z}{(z-a)(z-b)}$$

$$\text{find } f(0)=\lim_{z\to\infty}\frac{z}{(z-a)(z-b)}=0$$

$$f(1)=\lim_{z\to\infty}zF(z)=\lim_{z\to\infty}\frac{z^2}{(z-a)(z-b)}=1$$

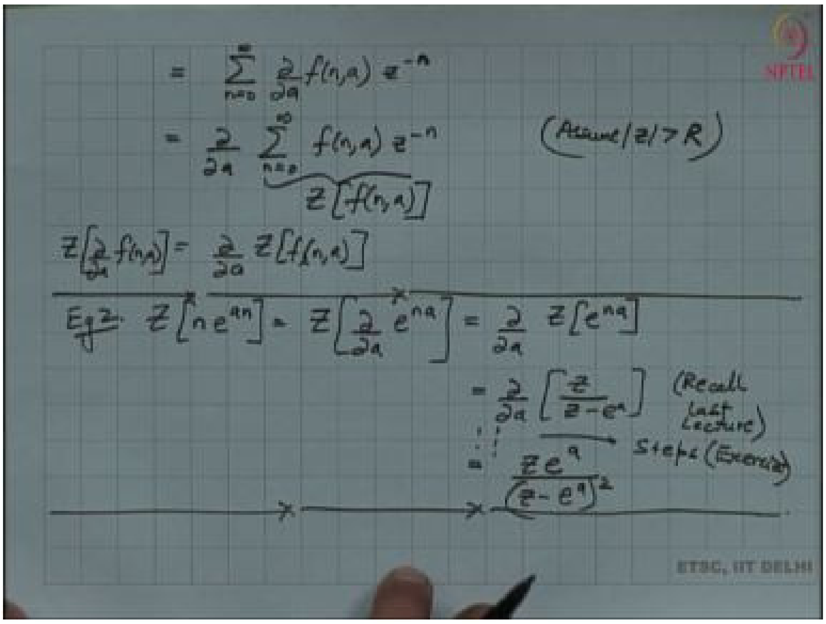
Theorem:

Z-transform of Partial Derivative:

$$z\left[\frac{\partial}{\partial a}f(n,a)\right]=\frac{\partial}{\partial a}z[f(n,a)]$$

Proof:

$$\text{LHS: } z\left[\frac{\partial}{\partial a}f(n,a)\right]=$$




$$= \sum_{n=0}^{\infty} \frac{\partial}{\partial a} f(n,a) z^{-n}$$

$$= \frac{\partial}{\partial a} \sum_{n=0}^{\infty} f(n,a) z^{-n}, \text{ Assume } /z/ > R$$

$$z\left[\frac{\partial}{\partial a} f(n,a)\right] = \frac{\partial}{\partial a} z[f(n,a)]$$

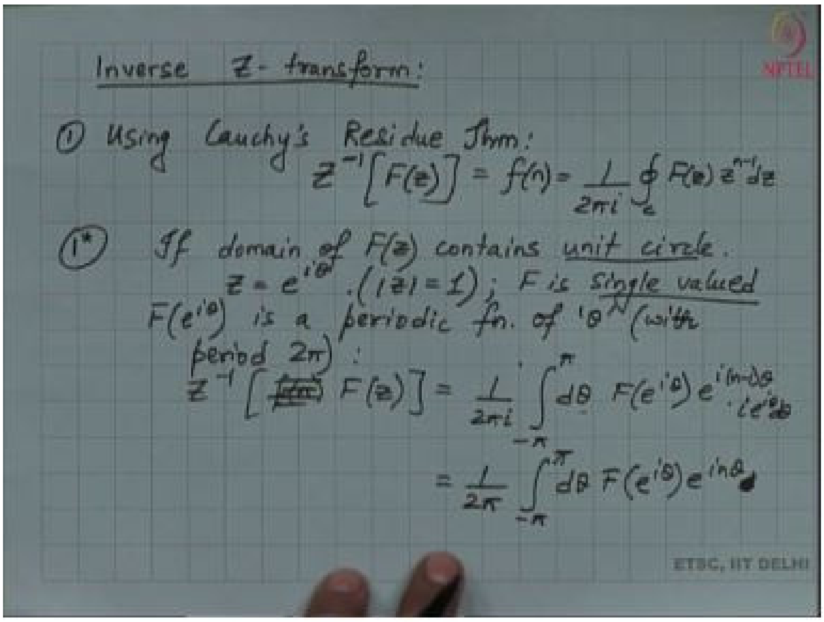
Example 2:

$$z\left[ne^{an}\right] = z\left[\frac{\partial}{\partial a} e^{na}\right] = \frac{\partial}{\partial a} z\left[e^{na}\right]$$

$$= \frac{\partial}{\partial a}\left[\frac{z}{z-e^{a}}\right]$$

$$= \frac{ze^{a}}{(z-e^{a})^2}$$

Inverse Z-transform




(1) using auchy's Residue Theorem:

$$z^{-1}[F(z)] = f(n) = \frac{1}{2\pi i} \oint_c f(z) z^{n-1} dz$$

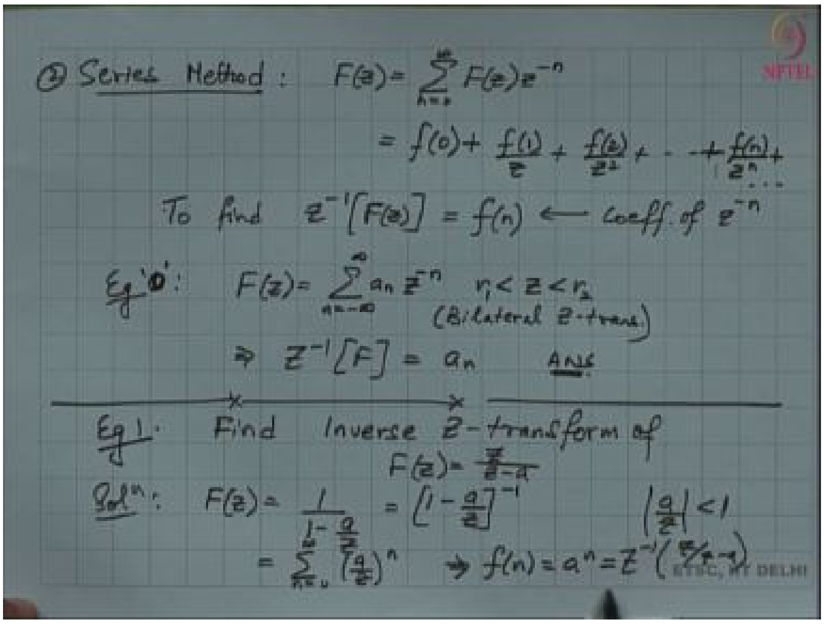
(1) If domain of $F(z)$ contains unit circle.

$z = e^{i\theta}, (/z/ = 1); F$ is single valued

$F\left(e^{i\theta}\right)$ is a periodic fn of $'\theta'$ (with period $2\pi$) :

$$\left[z^{-1} F(z)\right] = \frac{1}{2\pi i} \int_{-\pi}^{\pi} d\theta F\left(e^{i}\theta\right) e^{i(n-1)\theta} . ie^{\theta} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\pi} d\theta F\left(e^{i\theta}\right) e^{in\theta}$$



$$\text{(1) Series Method: } F(z) = \sum_{n=0}^{\infty} F(z) z^{-n}$$

$$= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots + \frac{f(n)}{z^n} + \cdot\cdot$$

$$\text{To find } z^{-1}(F(z)] = f(n) \leftarrow \text{ coefficient of } z^{-n}$$

$$\text{Example 0:} F(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n} \quad r_1 < z < r_2 \Rightarrow z^{-1}[F] = a_n$$

Example 1:

Find Inverse & -transform of

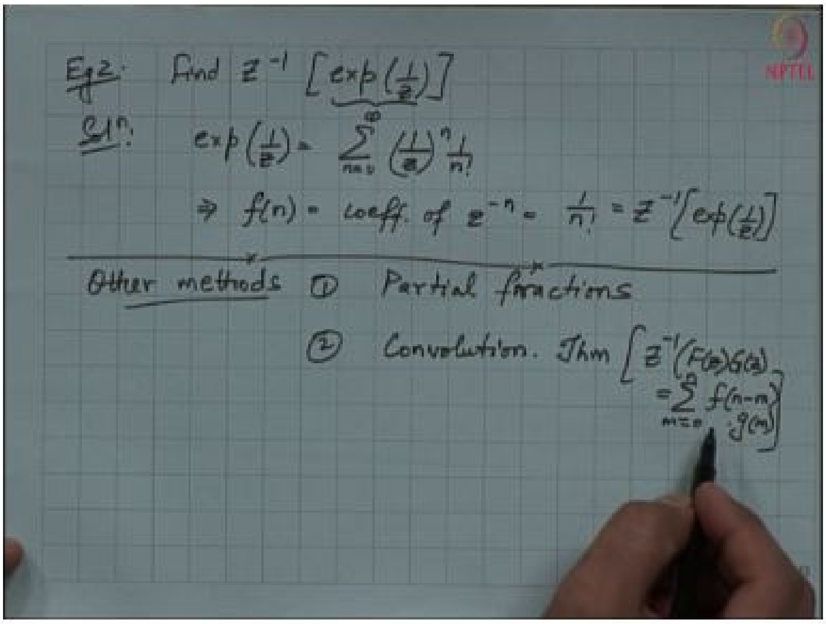
$F(z) = \frac{z}{z-a}$

$$F(z) = \frac{1}{1-\frac{a}{z}} = \left[1 - \frac{a}{z}\right]^{-1}$$

$$= \sum_{n=0}^{6} \left\{\frac{a}{z}\right)^n$$

$$\Rightarrow f(n) = a^n = z^{-1}\left(\frac{z}{z-a}\right)$$

then moving on let me show you another example on how to calculate the Z transform.




Example 2:

$$\text{Find } z^{-1}\left[\exp\left(\frac{1}{z}\right)\right]$$

Solution:

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \frac{1}{n!}$$

$$\Rightarrow f(n) = \text{ coeff of } z^{-n} = \frac{1}{n!} = z^{-1}\left(exp\left(\frac{1}{z}\right)\right)$$

Other methods:

(1) Partial functions

(2) Convolution. Them