Integral Transforms and Their Applications
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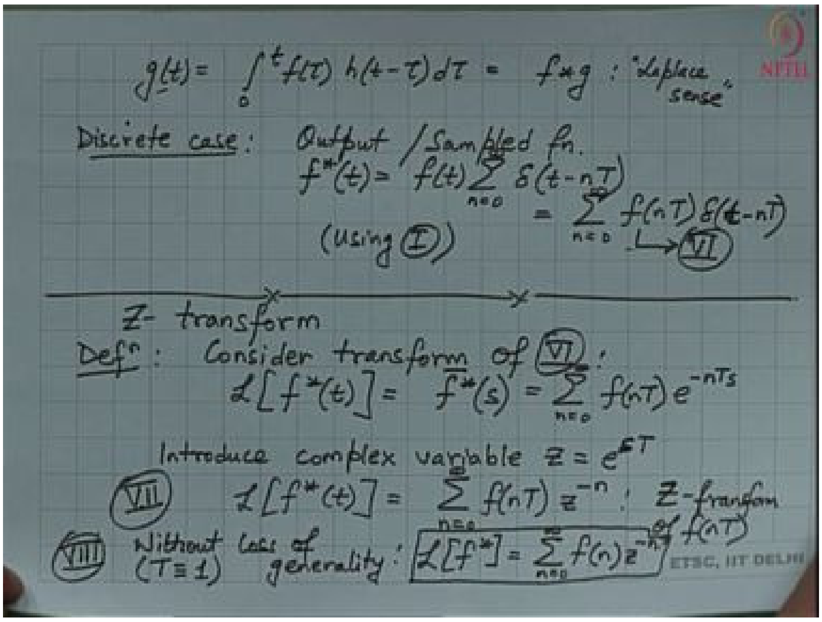
Lecture –44
Introduction to Z – transform Part- 02

So, moving on let me Introduce Z transform; so, now I have sufficient background. So, let me just introduce the definition of Z transform. So, consider the Laplace transform of the function. So, I am going to consider the Laplace transform. Well, so, Laplace transform I am going to denote the expression by expression number (VI). So, consider the Laplace transform of (VI).



So, I get that the Laplace transform of the sampled function $f^*(t)$:

$$\mathcal{L}\left[f^*(t)\right] = \bar{f}^*(s) = \sum_{n=0}^{\infty} f(nT)e^{-nT_s}$$

So, then we have that note that if I were to introduce let me introduce a new variable introduce the variable the complex variable $z = e^{sT}$.

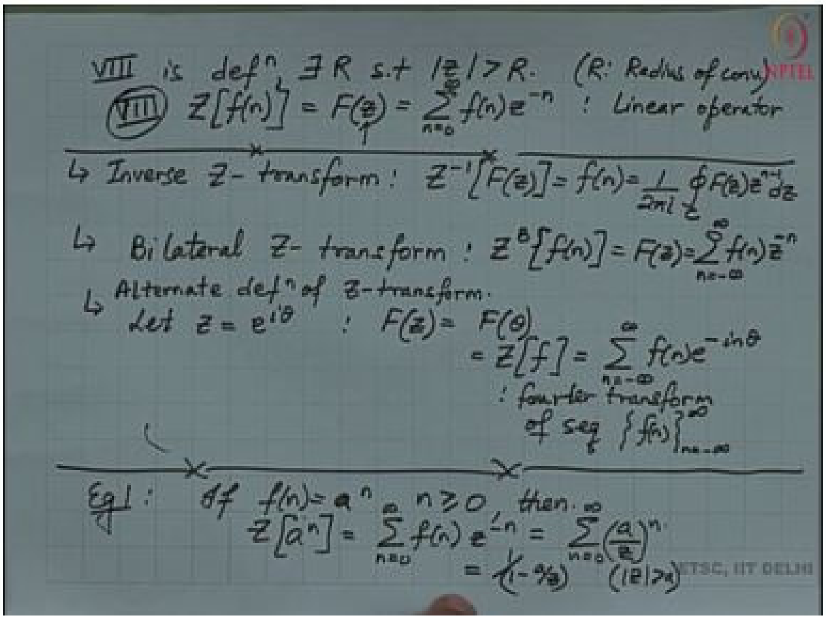
So, then my Laplace transform of the sampled well the output function or the output sampled function is,

$$\mathcal{L}\left[f^*(t)\right] = \sum_{n=0}^{\infty} f(nT)z^{-n} \qquad \text{...(VII) : Z-Transform of f(nT)}$$

I am going to slightly simplify this expression (VII) by saying without loss of generality;let me say that the Laplace transform of the sampled function is :

$$\mathcal{L}\left[f^*\right] = \sum_{n=0}^{\infty} f(n)z^{-n} \qquad (VIII)$$

So, what I have to done is in (VII) I have chosen T is identically equal to 1. So, my sample time points are at equidistant, but at unit spacing. So, my unit spacing so, my sampling is done at unit spacing. So, I am going to use this as my definition of Z transform.



So, I have defined (VIII). So, let me write down (VIII) once more. So, the Z transform of f(n).

$$Z[f(n)] = F(z) = \sum_{n=0}^{\infty} f(n)z^{-n} \qquad \text{: Linear Operator}$$

Now I can see that this is a linear it is a linear operator; it is a linear operator for with respect to the function f, so, it is a linear operator.

So, one is well (VIII) is defined for all radius of convergence it is defined for all values of R such that my complex variable z is greater than R. So, R is my radius of convergence in the complex plane. So, that is where my Z transform is going to be defined for that there is an R such that $|z| > R$. So, then once I have defined Z transform I also need to define the inverse transform.

So, let me introduce the inverse Z transform as follows.

$$Z^{-1}[F(z)] = f(n) = \frac{1}{2\pi i} \oint_c F(z) z^{n-1} dz$$

I also have another definition I have the definition of the so called bilateral Z transform.

$$Z^B[f(n)] = F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{-n}$$

So, let us now describe my Z transform in another way.Alternate definition of Z transform so,when I am working in polar coordinate. So, I have an alternate definition of Z transform.

$$\text{Let } z = e^{i\theta} \quad : \quad F(z) = F(\theta)$$

$$= Z[f] = \sum_{n=-\infty}^{\infty} f(n) e^{-in\theta}$$

So, this is we can see that this is the Fourier transform of the sequence of discrete functions the discrete sequence of f(n). I have introduced almost all the definitions that I need to work with Z transform. So, let us look at some examples in how to evaluate the Z transform.

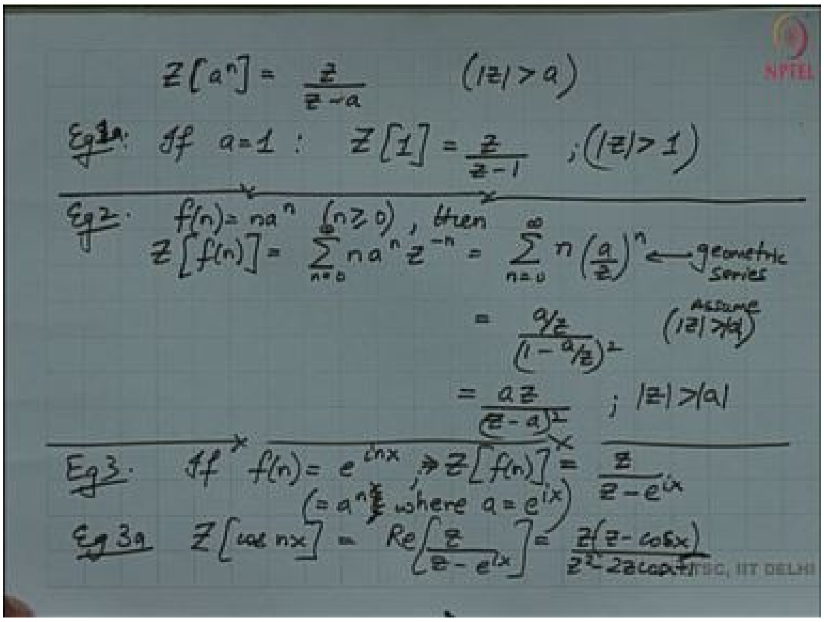
Example 1:If $f(n) = a^n$ then,Find Z-Transform.

Solution:

$$z\left[a^n\right] = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \frac{1}{(1 - a/z)} \qquad : |z| > a$$

$$Z\left[a^n\right] = \frac{z}{z - a} \qquad : |z| > a$$



Example 1(a): If $a = 1$ Solution:

$$z[1] = \frac{z}{z - 1} \qquad ; (|z| > 1)$$

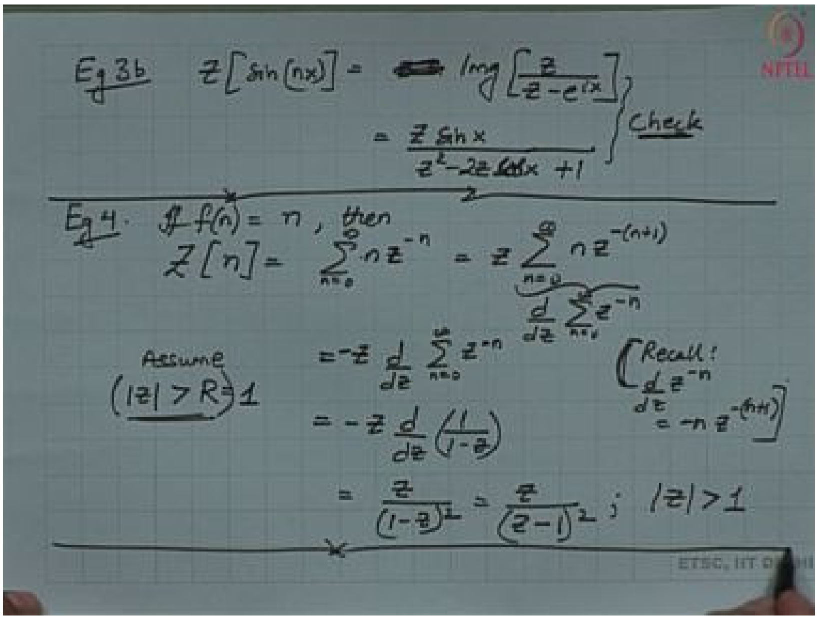
Example 2: $f(n) = na^n \quad (n \geqslant 0)$, then Solution:

$$Z[f(n)] = \sum_{n=0}^{\infty} n a^n z^{-n} = \sum_{n=0}^{\infty} n \left(\frac{a}{z}\right)^n$$

$$= \frac{a/z}{(1 - a/z)^2} \qquad : \text{Assume}(|z| > a)$$

$$= \frac{az}{(z - a)^2}; |z| > |a|$$

Example 3: If $f(n) = e^{inx}$. Solution:$Z[f(n)] = \frac{z}{z-e^{ix}}$
Example 3(a):$Z[\cos nx] = \text{Re}\left[\frac{z}{z-e^{ix}}\right] = \frac{z(z-\cos x)}{z^2-2z\cos x+1}$



Example 3(b): $\quad Z[\sin(nx)] = \text{Img}\left[\frac{z}{z-e^{ix}}\right]$

$$= \frac{z\sin x}{z^2 - 2z\cos x + 1}$$

Example 4:

$$\text{If} f(n) = n, \text{ then}$$

Solution:

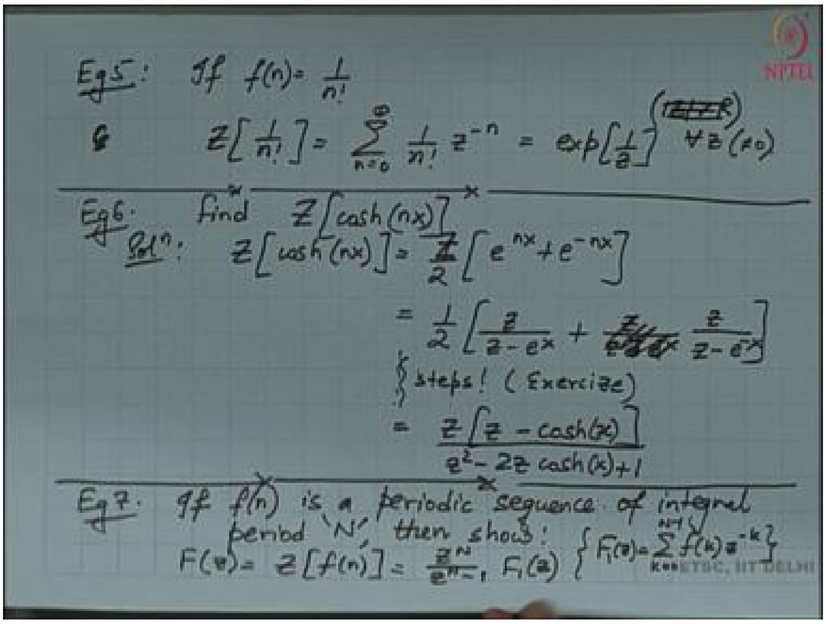
$$Z[n] = \sum_{n=0}^{\infty} nz^{-n} = z\sum_{n=0}^{\infty} nz^{-(n+1)}$$

$$= -z\frac{d}{dz}\sum_{n=0}^{\infty} z^{-n}$$

$$= -z\frac{d}{dz}\left(\frac{1}{1-z}\right)$$

$$= \frac{z}{(1-z)^2}$$

$$= \frac{z}{(z-1)^2}; |z| > 1$$



Example 5: If $f(n) = \frac{1}{n!}$
Solution:

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

using Taylor series ,

$$= \exp\left(\frac{1}{z}\right) \qquad : z \neq 0$$

Example 6: Find Z[cosh(nx)]
Solution:

$$Z[\cosh(nx)] = \frac{Z}{2}\left[e^{nx} + e^{-nx}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{x}} + \frac{z}{z - e^{-x}}\right]$$

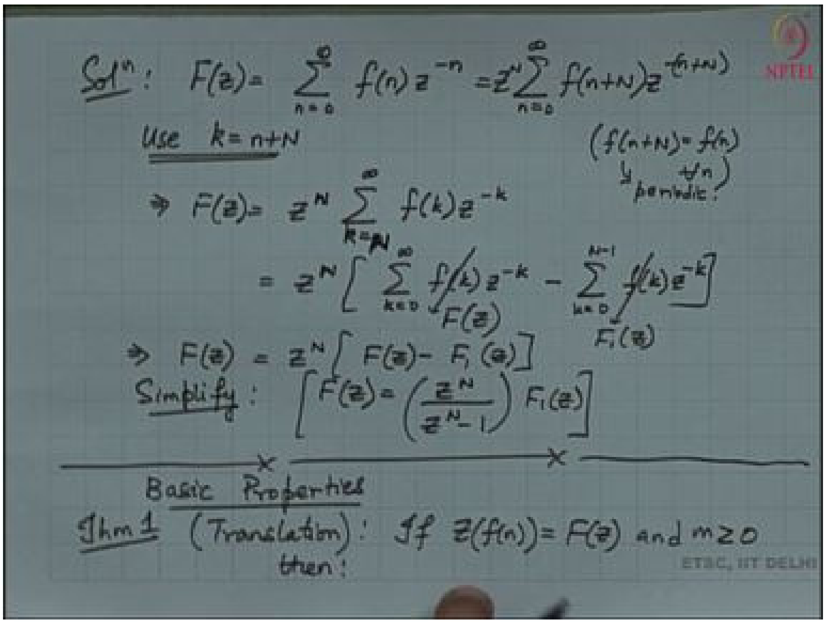
So, there are some steps which I have not shown. So, I asked the students to check that indeed from this expression I arrive at this particular expression here.

$$= \frac{z[z - \cosh(x)]}{z^2 - 2z\cosh(x) + 1}$$

Example 7:If f(n) is a periodic sequence of integral period of N then Show,

$$F(z) = Z[f(n)] = \frac{z^N}{z^N - 1} F_1(z)$$

$$\text{Where,} \quad F_1(z) = \sum_{k=1}^{N-1} f(k) z^{-k}$$



Solution:

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n} = Z^n \sum_{n=0}^{\infty} f(n+N) z^{-(n+n)}$$

Use k=n+N,

$$\Rightarrow F(z) = z^N \sum_{k=N}^{\infty} f(k) z^{-k}$$

$$= z^N \left[\sum_{k=0}^{N} f(k) z^{-k} - \sum_{k=0}^{N-1} f(k) z^{-k} \right]$$

$$\Rightarrow F(z) = z^N [F(z) - F_1(z)]$$

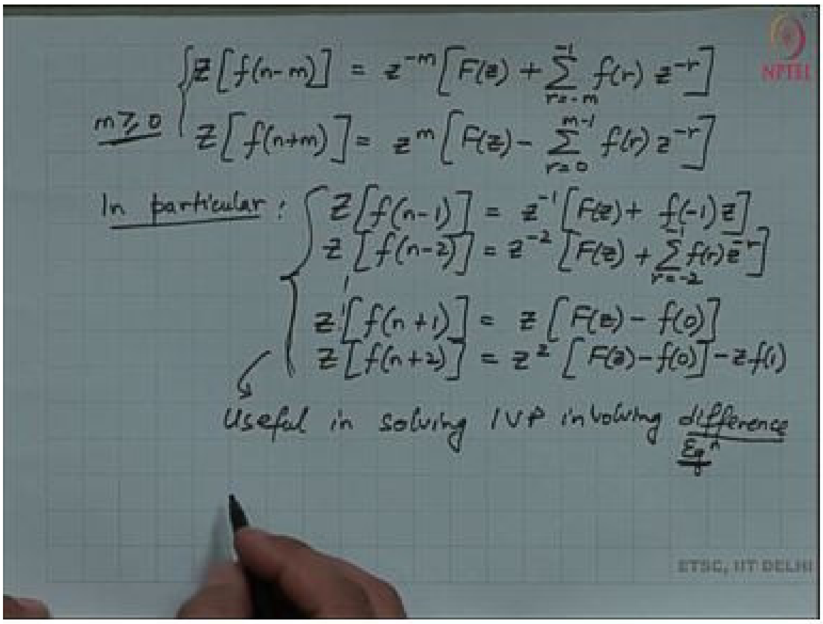
Simplify,

$$F(z) = \left(\frac{z^N}{z^N - 1} \right) F_1(z)$$

Basic Properties:
Theorem 1: Translation

$$\text{If } Z(f(n)) = F(z) \text{ and } m \geq 0$$

Well the result shows that there are 2 results the first result shows that ,



For, $m \geq 0$

$$Z[f(n-m)] = z^{-m}\left[F(z) + \sum_{r=-m}^{-1} f(r)z^{-r}\right]$$

$$Z[f(n+m)] = z^{m}\left[F(z) - \sum_{r=0}^{m-1} f(r)z^{-r}\right]$$

In Particular,let us look at the case of m =1.

$$Z[f(n-1)] = z^{-1}[F(z) + f(-1)z]$$

$$Z[f(n-2)] = z^{-2}\left[F(z) + \sum_{r=-2}^{-1} f(r)z^{-r}\right]$$

So, in a similar fashion I can write the Z transform of the function evaluated at n+1 and I get,

$$Z[f(n+1)] = z[F(z) - f(0)]$$

$$Z[f(n+2)] = z^{2}[F(z) - f(0)] - zf(1)$$

So, what I have done is why I have used I have outlined the specific case because we will see that especially these four cases and of course, the general expression as well they are quite useful; they are quite useful in solving some initial value problems involving difference equations. So, Z transforms are specially useful in solving difference equation because this is a discrete transform. So, the difference equations will involve discrete sums.Let us look at these two expressions and try to prove the result.