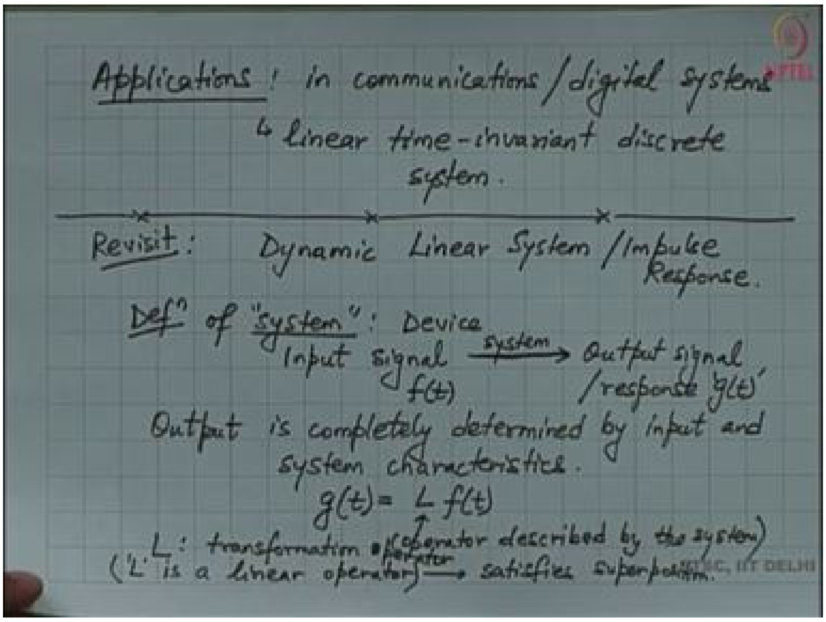
Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture –43
Introduction to Z – transform Part 1

Good afternoon everyone. So, today in this lecture, I am going to introduce the Z transform. As I have mentioned Z transforms are widely applicable transforms and specifically in when we have to evaluate some test cases in signal processing or some specific examples in electronics and communication systems.

Now, the most unique aspect of this transform is that this particular transform is defined as a discrete sum rather than the continuous integral that we have so far introduced in all my other previous transforms. So, let us continue our discussion on Z transforms.



So, as I have mentioned and as some of the students may already know Z transforms have wide ranging applications in communications and digital systems; in communications and digital systems and it is also applicable. Well the as I said the most unique part about the Z transform is that, it is going to be applied for linear time invariant discrete system.

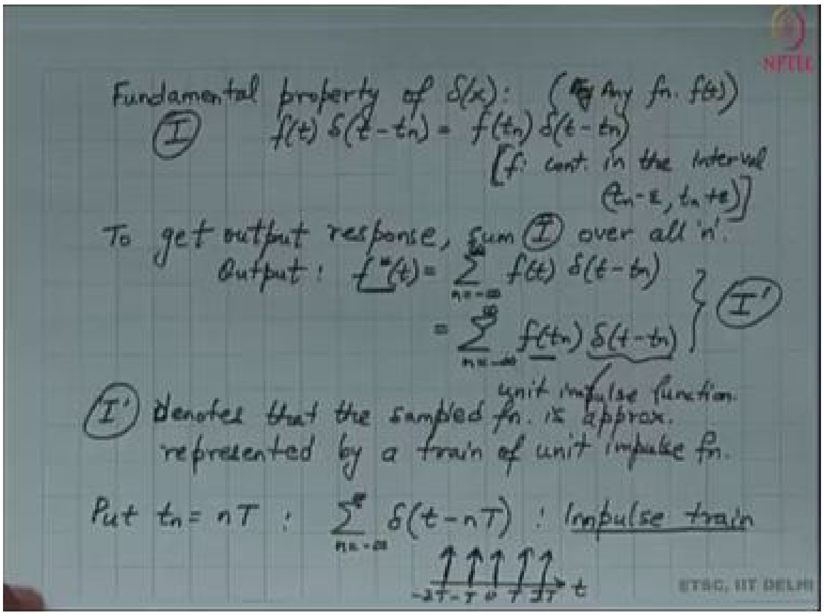
Linear time invariant discrete system as opposed to some of the other all the other transforms that we have seen in all our previous discussions in our earlier lectures. So, let me before I introduce the definition of Z transform, let me start with some background.

I am going to revisit my discussion on dynamic linear systems and introduced some more terms in greater details. So, I am I am revisiting my students who have seen through my lectures on Fourier transform, specifically in the example where I discuss Shannon sampling theorem must have encountered some of the terms, that I am going to go in slightly more detail now.

Dynamic linear system, impulse response impulse response function and so on. So, when I talk about a system; so, I am starting with the most basic definition. So, when I talk about a system, I say the system is like a black box. It is like a device where we feed in we feed in an input and let us say the input is in the form of a signal right. So, I call that signal to be the function f(t) and it goes through the system. So, the system is like a black box and what we get is an output signal or also known as the response function termed by this continuous function g(t).So, which means that my output my output is my output when I say output, I denote the output signal. The output is completely it is completely determined it is completely determined by my input signal and my system characteristics. So, my output is completely determined by my input and system characteristics. So, in terms of math notation, I say that my output g(t) is equal to L times the input $g(t) = Lf(t)$ where L is an operator which is described by the system. So, operator described by whatever system I am talking about described by the system.

I can write down my output signal as a relation in terms of the input signal more often than not like an ODE type of a of an equation ODE or PDE where L is the operator which is described by the specific system. For example, the system could be an LCR circuit or an LR circuit and so on.

I call this L as my transformation operator transformation operator. And now I also assume that L is a linear is a linear operator. So, L is a linear operator and L well this being and linear operator it is going to satisfy the principle of superposition. So, satisfies superposition principle.So,These are some of the things that we already know. So, I am just you know outlining, you know outlining some of these some of these definitions again to come to the definition of Z transform. So, then let us now introduce again reintroduce my idea of delta function.



So, because we are going to described my describe my continuous function using a discrete sum of delta functions. So, again let us come to the delta functions itself. So, let me start by outlining some fundamental properties of delta function, I see that I have this following relation that is that holds true.

Fundamental Property of $\delta(x)$:

(I).

$$f(t)\delta\left(t-t_n\right)=f\left(t_n\right)\delta\left(t-t_n\right)$$

So, I well all I need to make sure is that f(t) is continuous in the interval around the time point t_n.

So, then well, we will see that almost all that the fields involving sampling theory or you know pulse modulation systems or in fact, including systems including feedback like the filters or ECG cardiogram, they all then they all use this fundamental property of delta function. So, then coming let me just call this property by 1, because I will be using this time and again. So, if I were to get the output function or the output signal, I use this property of delta function to get the output response sum (I) over all n right. So, whatever time points that I have the information ,I sum it over all such time points.

Output=

$$f^*(t) = \sum_{n=-\infty}^{\infty} f(t)\delta\left(t - t_n\right)$$

$$= \sum_{n=-\infty}^{\infty} f\left(t_n\right)\delta\left(t - t_n\right) \qquad ....(\text{I'})$$

I also call this as well from now on we see that this output from this sum,can be written as the sum of the input functions f_n summed over discreetly over specific time points t_n.

So, if my input signals were one were unity then my discrete signals were completely represented by this delta function. So, which means for input equal to 1 the delta functions denote my input at time points t_n. So, I call these delta functions as my unit impulse function. why? Because it has well it is an impulsive function, well it gives an impulse at each time point $t = t_n$ for the function with an amplitude equal to 1. So, then so let me call this expression this summation expression by let us say (I').

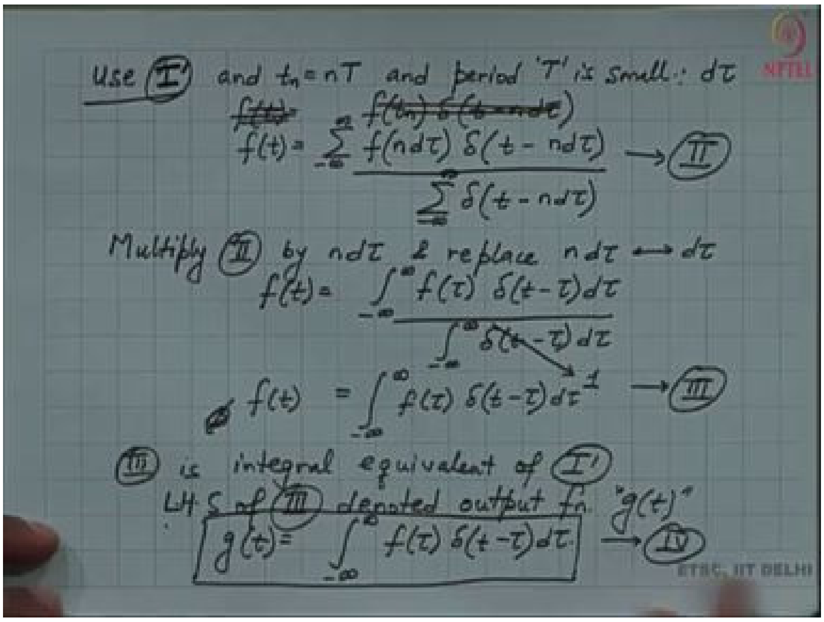
So,what does (I') denotes? So, (I') denotes that the sampled function the is approximately represented by a train of unit impulse function. which is the delta functions. Now let us now further make another approximation let us say that I am going to use my time points put t_n to be equally spaced.

Let us say my time points are spaced equally with interval capital T. So, if I do that then in this particular case the series and from $-\infty$ to ∞ is also known as the impulse train this is in this specific case I denote this summation as my impulse train. So, all these terms are quite prevalent in signal processing. So, what do I mean by impulse train? Which means that suppose if I were to measure these signals at these time points t_n which are equally spaced. So,let us say this is at $0, -T, -2T, T, 2T$ and so on.

Put, $t_n = nT$,

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad ;\text{Impulse Train}$$

I see that my impulse functions have an integral value equal to 1. So, they are all equal and they arrive at equal intervals of time. So, this is a unit impulse function at equidistant point of time known as the impulse train.



So, moving on I see that,again I am going to use I and I am going to use the fact that my time points tn are equally spaced to arrive at and also further I am going to use that my period T is small.

Let us say I denote the small period as $d\tau$.if I use my expression I, I come to the conclusion that,

$$f(t) = \frac{\sum_{-\infty}^{\infty} f(nd\tau)\delta(t - nd\tau)}{\sum_{-\infty}^{\infty} \delta(t - nd\tau)} \qquad ....(II)$$

So, then next I am going to let me call this as my expression (II). So, I am going to multiply (II) by $nd\tau$ and I am going to replace $nd\tau$ by the variable $d\tau$.

So, what I am doing is I am notice that this is n summation over n infinite domain. So, this is discrete sum; so, discrete sum written in the form of a series. So, well the series written in discrete values of the function evaluated at discrete time points. So, what I am trying to do is that I am slowly going to change this summation to an integral. So, when we do that I get I get my sampled function to be equal to integral $-\infty$ to ∞.

I have already made by my points of sampling $nd\tau$ to be sufficiently small. So, that my summation can be safely changed into an integral and I get that my function f(t) can be written in this integral representation ,

$$f(t) = \frac{\int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau}{\int_{-\infty}^{\infty} \delta(t - \tau)d\tau}$$

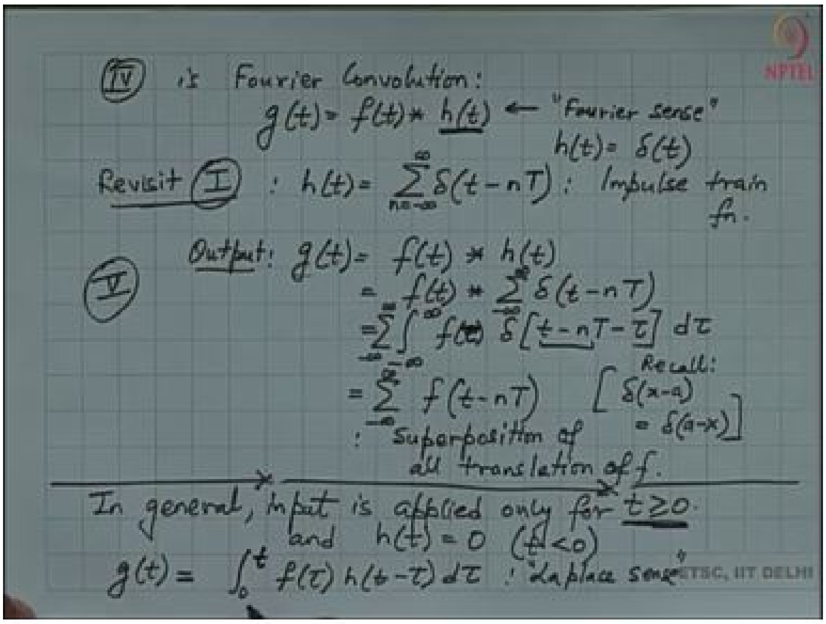
$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau \qquad ...(III)$$

So, (III) is the integral equivalent of (I'). So, that (I') was the discrete case and (III) is the integral equivalent. So, then we see that so, if we look at (III) carefully we see that if my left hand side can be denoted as my output function.Let us say that the LHS of (III) if I denote it by the output function. Let us call that this is g(t), then the output function can be represented as this convolution of the input signal with this delta function.

So, what I am saying is the following. So, if the LHS of (III) can be represented by the output function let us denote it by another new function g(t). So, my g(t) is the output of the system and my f(t) is the input of the system which is what the convention that I started with. So, in that case my g(t) is:

$$g(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau \qquad ....(\text{IV})$$

So, what this relation shows that, if I have a system where my individual signals are represented by these unit impulse functions then my output can be represented from the input signal by this following integral and then I can see that this particular integral is nothing, but the Fourier convolution integral.



So, (IV) is the Fourier the Fourier convolution right. So, what I have is that the output is :

$$g(t) = f(t) * h(t) \qquad : \text{Fourier Sense}$$

So, far we have described convolution in many different ways related to each of the transforms. So, this is in the Fourier sense.

So, then now so again coming back to, if I revisit my expression (I) in my previous slide, and I take my function h(t) to be,

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \qquad : \text{Impulse Train Function}$$

So, then what I have is that my output. If I were to represent my output my output g(t),

$$g(t) = f(t) * h(t)$$

$$= f(t) * \sum_{-\infty}^{\infty} \delta(t-nT)$$

$$= \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)\delta[t - nT - \tau]d\tau$$

$$= \sum_{-\infty}^{\infty} f(t - nT) \qquad ...(V)$$

So, here I have used the fact that,

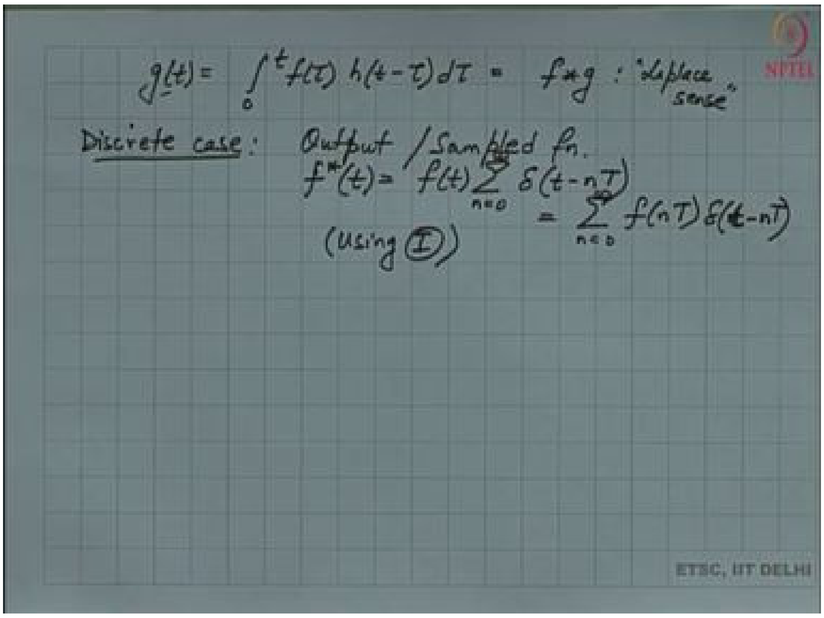
$$\delta(x - a) = \delta(a - x)$$

So, this particular expression let me call this as (V), I have shown here is that the output of the of any signal can be represented by an infinite sum at discrete points of time of infinite discrete sum of the input at regular points of time interval. So, the output is the superposition, it is the superposition of all translations of f. So, which means which means that my output can be represented in the term of a discrete sum of input.

In General, Input is applied only for $t \geq 0$ and $h(t) = 0$ for $t < 0$,then,

$$g(t) = \int_0^t f(\tau)h(t - \tau)d\tau \qquad \text{: Laplace Sense}$$

So, our output is the convolution of the input, but this is in the Laplace sense not in the Fourier sense. So, this is in the Laplace sense of convolution why because I have completely discarded the negative half of the integral. So, moving on if I were to write this output so, let me write this expression again.



$$g(t) = \int_0^t f(t)h(t - \tau)d\tau = f{*}g$$

So, the convolution in the Laplace sense Laplace transform sense. So, if I were to write this output this integral form in the discrete in the discrete notation, then in that case. So, in the discrete notation the discrete case my output or my sampled function my output or my sampled function can be shown to be let me call this I am going to distinguish it from g(t) because that is the continuous case.

Discrete Case:

$$f^*(t) = f(t) \sum_{n=0}^{\infty} \delta\left(t - nT\right)$$

$$= \sum_{n=0}^{\infty} f(nT)\delta(t - nT)$$

So, using (I) this is the discrete case of the above expression in the integral form. So, if we have a signal which is only measured at well at real times which will be positive then we will use the following discrete summation to represent the output signal.So far I have just created I have provided you some of the basic backgrounds and from now on I am going to describe my Z transfer using these discrete summation. So, moving on.