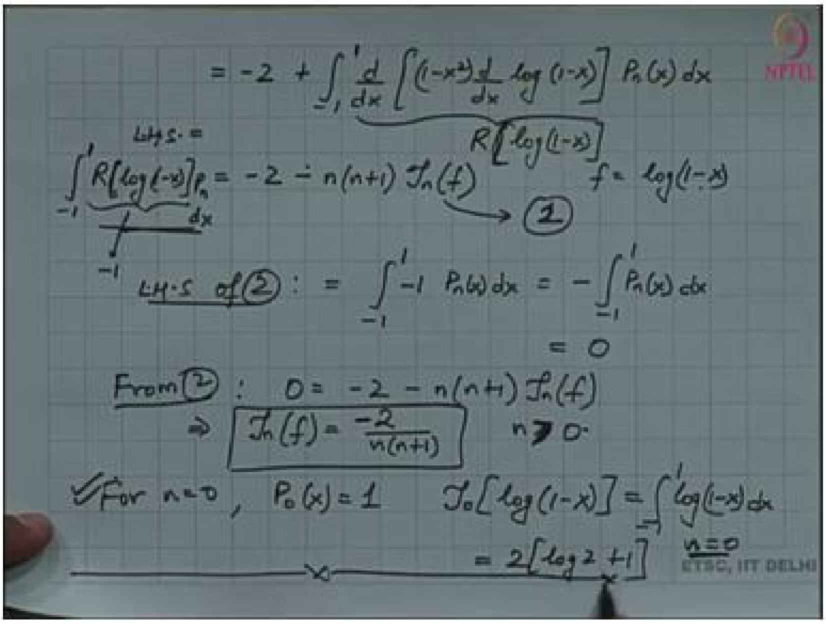
Integral Transforms and Their Applications

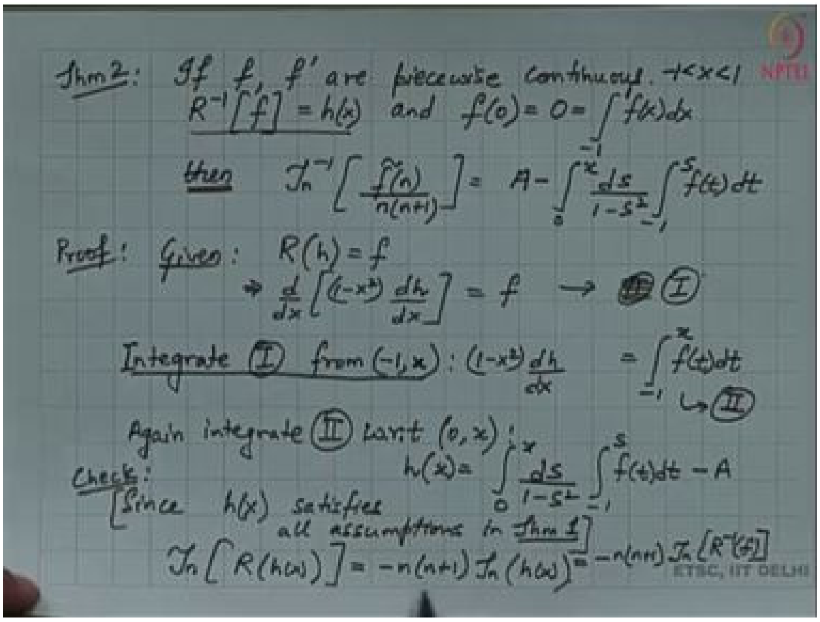
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi

Lecture –42

Introduction to Legendre Transform Part - 03



So, moving on; I have another result in the form of a theorem. So, this example is done here, I move on to the next result.



Theorem 2:if I am given f and f' are piece-wise continuous and in this limit range $-1 < x < 1$ and I am given that ,

$$R^{-1}[f] = h(x) \text{ and } f(0) = 0 = \int_{-1}^{1} f(x)dx$$

then,

$$\mathcal{J}_n^{-1}\left[\frac{\tilde{f}(n)}{n(n+1)}\right] = A - \int_0^x \frac{ds}{1-s^2}\int_{-1}^s f(t)dt$$

Proof:Given,

$$R(h) = f$$

$$\Rightarrow \frac{d}{dx}\left[\left(1-x^2\right)\frac{dh}{dx}\right] = f............(I)$$

I am going to integrate my expression (I) from - 1 to x and let us see what happens,

$$\left(1-x^2\right)\frac{dh}{dx} = \int_{-1}^x f(t)dt......(II)$$

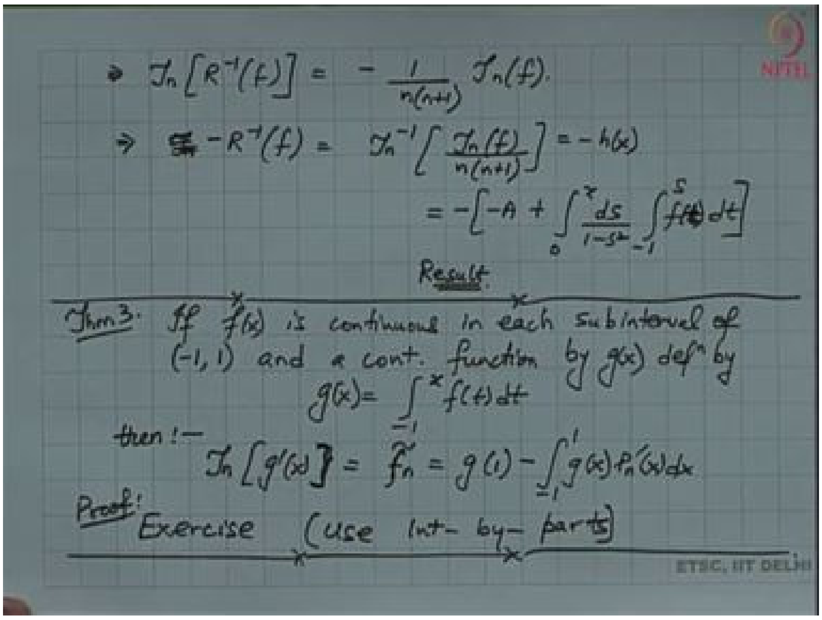
Again Integrate expression (II) from 0 to x:

$$h(x) = \int_0^x \frac{ds}{1-s^2}\int_{-1}^s f(t)dt - A$$

Check, Since h(x) satisfies all assumptions in Theorem (I),
then,

$$\mathcal{J}_n[R(h(x))] = -n(n+1)\mathcal{J}_n(h(x)) = -n(n+1)\mathcal{J}_\backslash[R^{-1}(f)]$$

$$\mathcal{J}_n\left[R^{-1}(f)\right] = -\frac{1}{n(n+1)}\mathcal{J}_n(f)$$



$$-R^{-1}(f) = \mathcal{J}_n^{-1}\left[\frac{\mathcal{J}_n(f)}{n(n+1)}\right] = -h(x)$$

$$= -\left[-A + \int_0^x \frac{ds}{1-s^2}\int_{-1}^5 f(t)dt\right]$$

Theorem 3:So, then I have one more result in the another theorem, I stated in in the form of another theorem. It tells me that if I am given a function f which is continuous function f which is continuous in each sub interval in each sub interval of -1 to 1. So, it is piece-wise continuous and I define another continuous function, I define another continuous function by g(x) defined by defined by the integral of f.

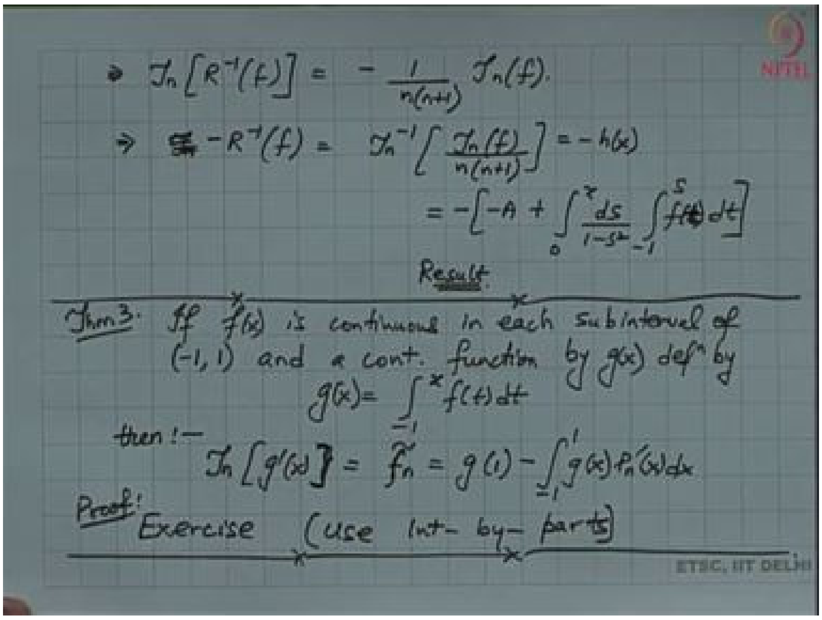
$$g(x) = \int_{-1}^{x} f(t)dt$$

then,

$$\mathcal{J}_n\left[g'(x)\right] = \tilde{f}_n = g(1) - \int_{-1}^{1} g(x)f_n'(x)dx$$

Proof:Now, to show this result it is quite the form of the result shows that we have to use integration by parts. So, I would rather leave this the proof of this theorem to the students to see that this is indeed the case. So, you need to use integration by parts to come to this conclusion given by the theorem.

So, then let me state another result.



Theorem 4:Let me state a result again in the form of another theorem namely the result on convolution of two Laplace; well the convolution results on Legendre transform. So, it says that I am given the Legendre transform of two functions.

$$\text{If } \mathcal{J}_\backslash(f) = \tilde{f}(n)$$

$$\text{If } \mathcal{J}_\backslash(g) = \tilde{g}(n) \quad \text{then, } \mathcal{J}_\backslash(f * g) = \tilde{f}(n)\tilde{g}(n)$$

Definition:

$$f * g = h(x) = \frac{1}{\pi}\int_0^{\pi} f(\,\cos\,\mu)\sin\mu d\mu \int_0^{\pi} g(\cos\lambda)d\beta$$

where,

$$x = \cos v$$
$$\cos\lambda = \cos\mu\cos v + \sin\mu\sin v\cos\beta$$

Application:

let us look at one application of Legendre transforms. As I said earlier Legendre transform has lot of applications in potential functions and in boundary value problems involving Laplacian. So, I am going to describe an example in a boundary value problem involving Laplacian. So, Laplace in the a boundary value problem in Laplace equation.So, the problem says solve the Dirichlet problem solve the Dirichlet problem for the potential function solve the Dirichlet problem for the potential function $u(r,\theta)$ inside a unit sphere inside a unit sphere, given by well inside a unit sphere and the potential satisfies the following equation it is the Laplacian in the r,θ frame.

So, I have so, note the way how I have written the Laplace equation. So, I have the regular Laplacian with respect to r and then I use my transformed;

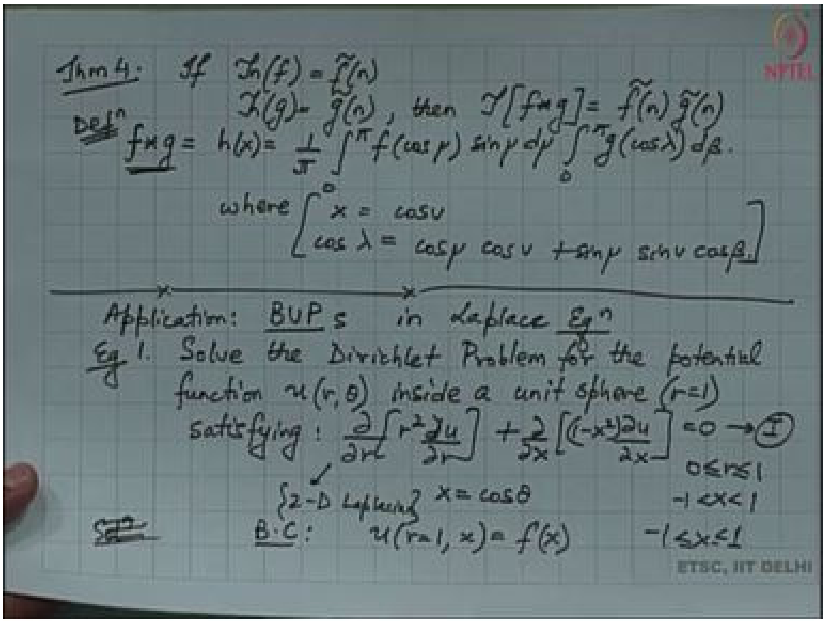
$$\frac{\partial}{\partial r}\left[r^2\frac{\partial u}{\partial r}\right] + \frac{\partial}{\partial x}\left[\left[(1-x^2)\right]\frac{\partial u}{\partial x}\right] = 0.....(I)$$

where we have taken ,

$$0 \leqslant r \leqslant 1$$
$$-1 < x < 1$$

Boundary Conditions are,

$$u(r=1,x) = f(x)$$



Apply Legendree's Transform, Based on Theorem (I)

$$r^2\frac{d\tilde{u}}{dr^2} + 2r\frac{d\tilde{u}}{dr} - n(n+1)\tilde{u} = 0...(I')$$

Boundary Condition,

$$\tilde{u}(1,n) = \tilde{f}(n)$$

(I') has two roots,

$$\tilde{u}_1 = Ar^n$$
$$\tilde{u}_2 = Br^{-(n+1)}$$

I know that the solution is bounded,$0 \leq r \leq 1$

$$\tilde{u}(r,n) = \tilde{f}(n)r^n$$

Apply Inverse Transform,

$$u(r,n) = \mathcal{J}_n^{-1}\left[\{r^n\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)\tilde{f}(n)r^n P_n(x) \quad 0 < r < 1$$

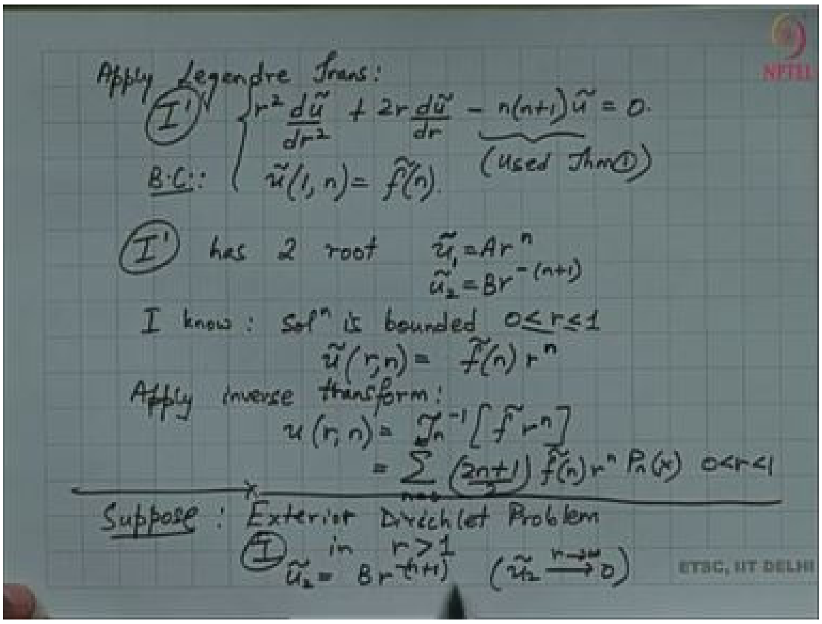
But, before I end this this example I just also want to highlight that, suppose I want to solve the problem suppose I want to solve the complement of this problem also known as the exterior Dirichlet problem right.

So, suppose I am given and I am asked to solve the exterior Dirichlet problem, that is I am I that is I am ask to solve I in the domain $r > 1$. So, in that case I can see that from well all the steps will follow all the way up to I', but then when I get to this point where we have 2 roots. So, when I have 2 roots I see that only the second root is going to make sense in the range from r for $r > 1$ where, the solution is going to be bounded at ∞. So, use so I am going to use ,

$$\widetilde{u_2} = Br^{(-n+1)}$$

I can see that u 2 is bounded when $r \to \infty$. So, r equal to 0 is not in the domain, but r tending to ∞ is.

$$u(r,x) = \sum_{n=0}^{\infty}\left(\frac{\tilde{f}}{r^{n+1}}\right)\left(\frac{2n+1}{2}\right)P_n(x) \qquad r > 1;$$



So, that completes that completes the discussion on that completes the discussion on Legendre transforms and the students are also requested to see more examples specially the handouts and the assignments that I am going to provide.

Now, in the next lecture, I am going to introduce yet another new transform known as the Z-transform. So, people specially working in signal processing and in you know electronics and communications, they must have heard a little bit about z transforms. The most you know unique part about Z-transform is this is one of the few transforms which is defined as a discrete sum. So, so far I have only introduced transforms which are defined in terms of an integral. So, in the next lecture the Z-transform which will be introduced we will see that it is a discrete sum so. Thank you very much; thank you for listening.