Integral Transforms and Their Applications
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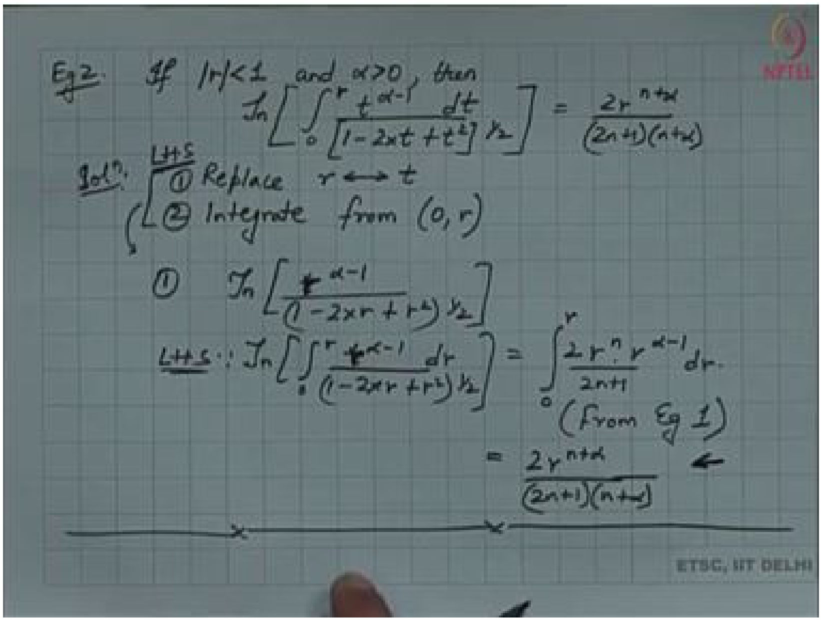
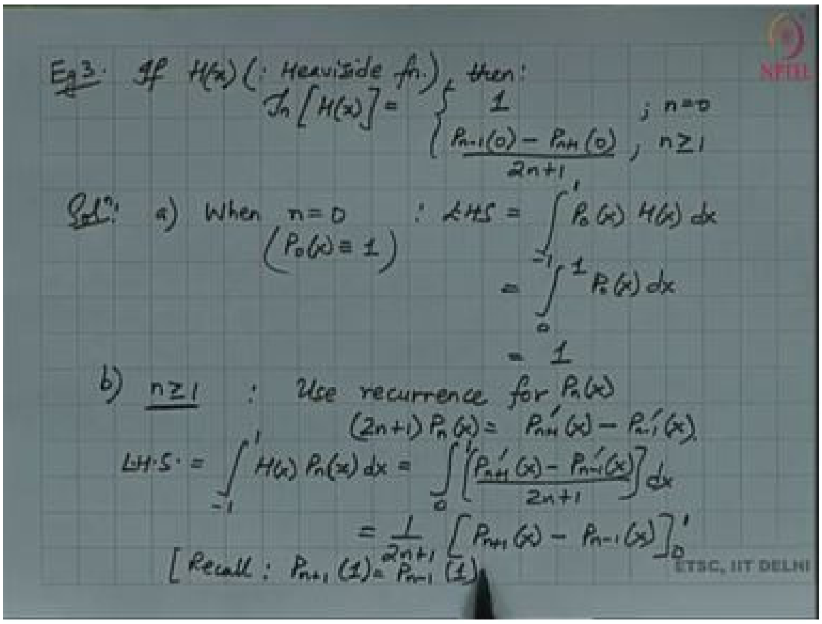
Lecture –41
Introduction to Legendre Transform Part - 02



So, then I have another example.



Example 3:The third example says let us say I have the Heaviside function. So, H is the Heaviside function, I am given that the Legendre transform of the Heaviside function is given by :

$$\mathcal{J}_n[H(x)] = \begin{cases} 1 & , \text{ n=0} \\ \frac{P_{n-1}(0)-P_{n+1}(0)}{2n+1}, & n \geq 1 \end{cases}$$

Solution:a)When n=0, then$P_0(x) = 1$;

$$\therefore LHS = \int_{-1}^{1} P_0(x)H(x)dx$$

$$= \int_{0}^{1} P_0(x)dx = 1$$

b)$n \geq 1$ use recurrence of $P_n(x)$,

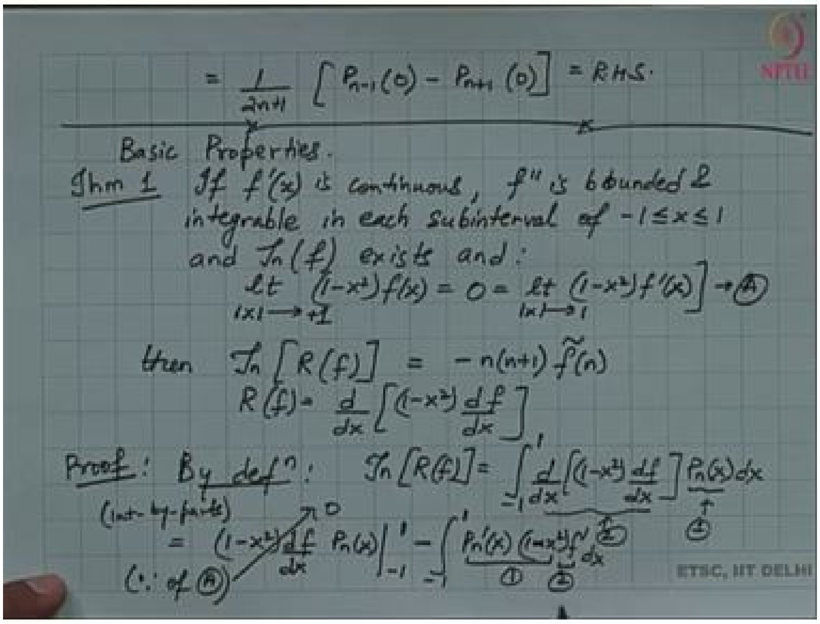
$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$

$$L.H \cdot S = \int_{-1}^{1} H(x)P_n(x)dx = \int_{0}^{1} \left(\frac{P'_{n+1}(x) - P'_{n-1}(x)}{2n+1} \right) dx$$

$$= \frac{1}{2n+1} \left[P_{n+1}(x) - P_{n-1}(x) \right]_0^1$$

$$\text{Recall: } P_{n+1}(1) = P_{n-1}(1)$$

$$= \frac{1}{2n+1} \left[P_{n-1}(0) - P_{n+1}(0) \right] = RHS$$



Basic Properties:

Theorem 1: Theorem says if I have a function such that its derivative is continuous and and $f''$is bounded it is bounded and integrable bounded and integrable in each sub interval it is bounded and integrable in each sub interval of the value from $-1 < x < 1$.

I am given that the transform of f exists and further I am given this following relation,

$$\lim_{|x| \to +1} \left(1 - x^2\right) f(x) = 0 = \lim_{|x| \to} 1 \left(1 - x^2\right) f'(x) \to (A)$$

$$\text{then } \mathcal{J}_n[R(f)] = -n(n+1)\tilde{f}(n)$$

$$R(f) = \frac{d}{dx}\left[\left(1-x^2\right)\frac{df}{dx}\right]$$

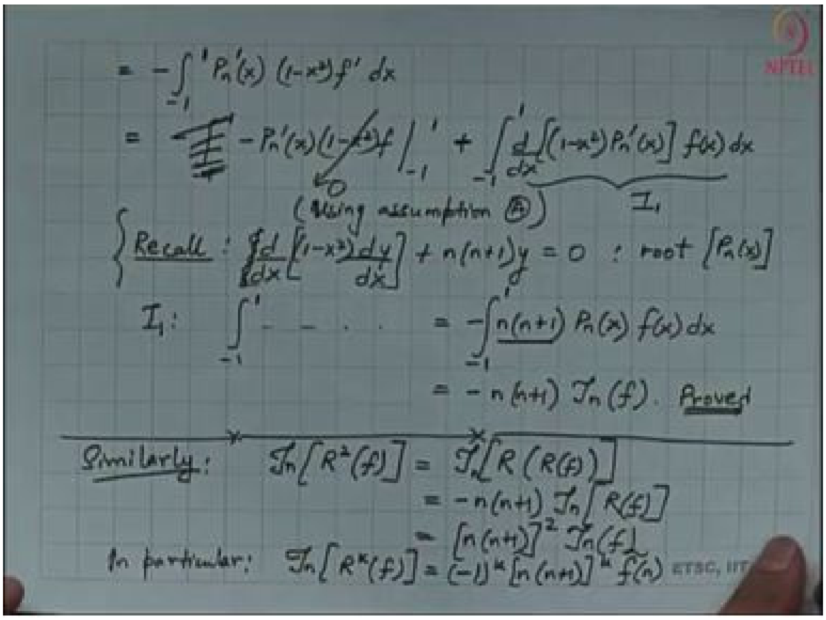
Proof: By Definition,

$$\mathcal{J}_n[R(f)] = \int_{-1}^{1} \frac{d}{dx}\left[\left(1-x^2\right)\frac{df}{dx}\right] P_n(x)dx$$

Solving using Integration by parts,

$$= \left(1-x^2\right)\frac{df}{dx}P_n(x)\Big|_{-1}^{1} - \int_{-1}^{1} P_n'(x)(1-x^2)f'dx$$

So, what I have is the following. What I have is, well we see that the way to proceed is to start integrating using integration by parts and the natural choice of the first function will be this one the Legendre polynomial and then this I choose is as my second function.So, then what I get is that this is also equal to, so using integration by parts I get that the following expression:



$$= -\int_{-1}^{1} P_n'(x)\left(1-x^2\right)f'dx$$

$$= \; -(1-x^2)f \; P_n'(x)\big|_{-1}^{1} - \int_{-1}^{1} \frac{d}{dx}P_n'(x)(1-x^2)f(x)dx$$

So, I see that this well the first one again is 0 because I have used using my assumption (A), right. So, this first part again vanishes due to my the second of the assumption within (A), and then the second is an integral, we see that this particular integral can be solved. So, by noting what is my Legendre equation? So, recall my Legendre equation we call that that I have the following equation that is satisfied by the Legendre polynomial.

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+n(n+1)y=0 \qquad \text{roots : } [P_n(x)]$$

Second part of above integral which is denoted by I_1 is given as,

$$\begin{aligned} I_1 &= -\int_{-1}^{1} n(n+1)P_n(x)f(x)dx \\ &= -n(n+1)\mathcal{J}_n(f) \end{aligned}$$

Similarly,

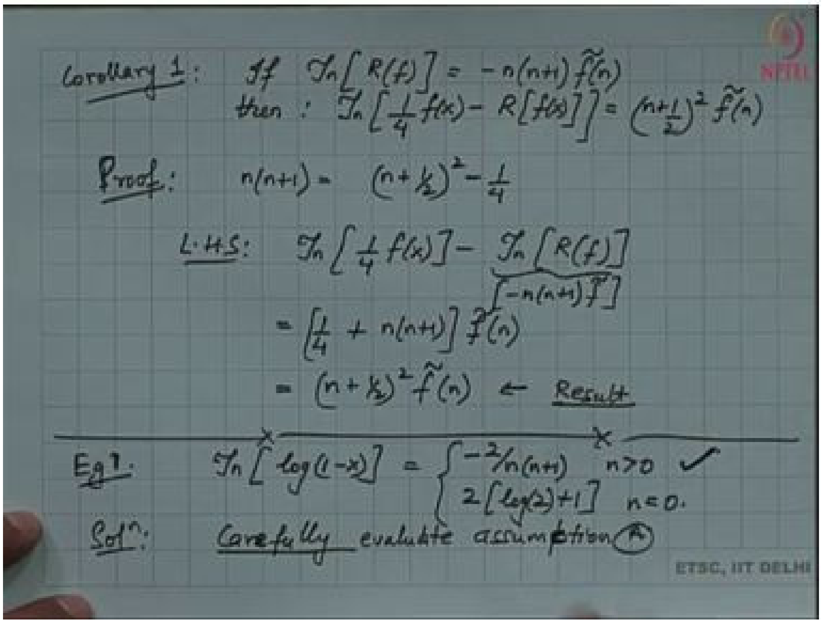
$$\mathcal{J}_n\left[R^2(f)\right] = \mathcal{J}_n[R(R(f))]$$

$$= -n(n+1)\mathcal{J}_n[R(f)]$$

$$= -[n(n+1)]^2\mathcal{J}_n(f)$$

In Particular,

$$\mathcal{J}_n\left[R^k(f)\right] = (-1)^k[n(n+1)]^k f(n)$$



Corollary 1:

$$\text{If } \mathcal{J}_n[R(f)] = -n(n+1)\tilde{f}(n) \quad \text{then,}$$

$$\mathcal{J}_n\left[\frac{1}{4}f(x)-R[f(x)]\right] = \left(n+\frac{1}{2}\right)^2\tilde{f}(n)$$

Proof:

$$n(n+1) = (n+1/2)^2 - \frac{1}{4}$$

$$\text{LHS: } \mathcal{J}_n\left[\frac{1}{4}f(x)\right] - \mathcal{J}_n[R(f)]$$

$$= \left[\frac{1}{4} + n(n+1)\right]\tilde{f}(n)$$

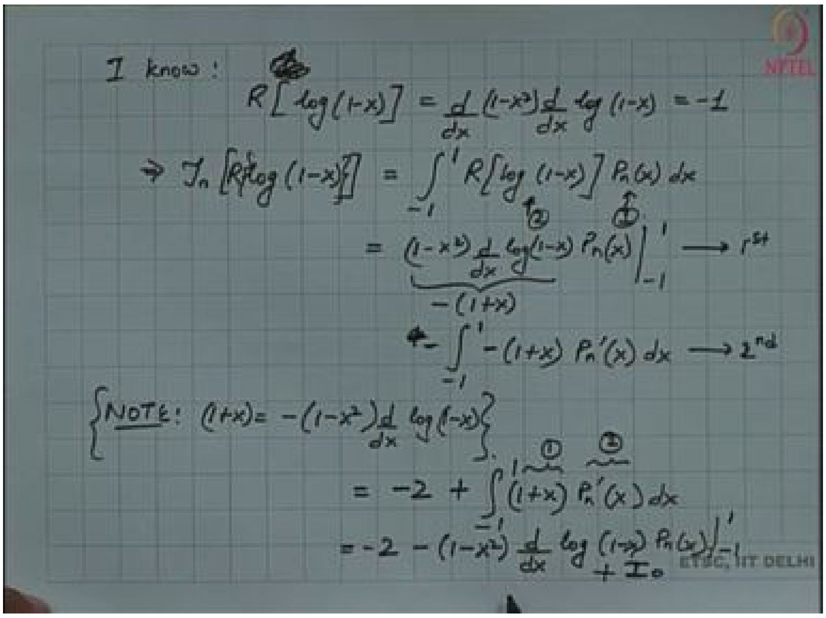
$$= \left(n+\frac{1}{2}\right)^2 \tilde{f}(n)$$

So, moving on I have more results, well I have one example now to use the theorem above. So, the example says I have to show that

Example 1:

$$\mathcal{J}_n[\log(1-x)] = \begin{cases} \frac{-2}{n(n+1)} & n > 0 \\ 2[\log(2)+1] & n = 0 \end{cases}$$

Solution:So, this is the result,I have to carefully evaluate assumption A. So, the limit may or may not exist depending on, well we have to take the limit carefully in order that assumption A in our theorem is satisfied, carefully evaluate assumption A. So, under the careful evaluation I will go ahead and utilise my theorem 1, namely this expression here at the top. So, let us look at the n positive case first because the general case is more involved.



So, I have that I have so I know the following.

$$R[\log(1-x)] = \frac{d}{dx}(1-x^2)\frac{d}{dx}\lg(1-x) = -1$$

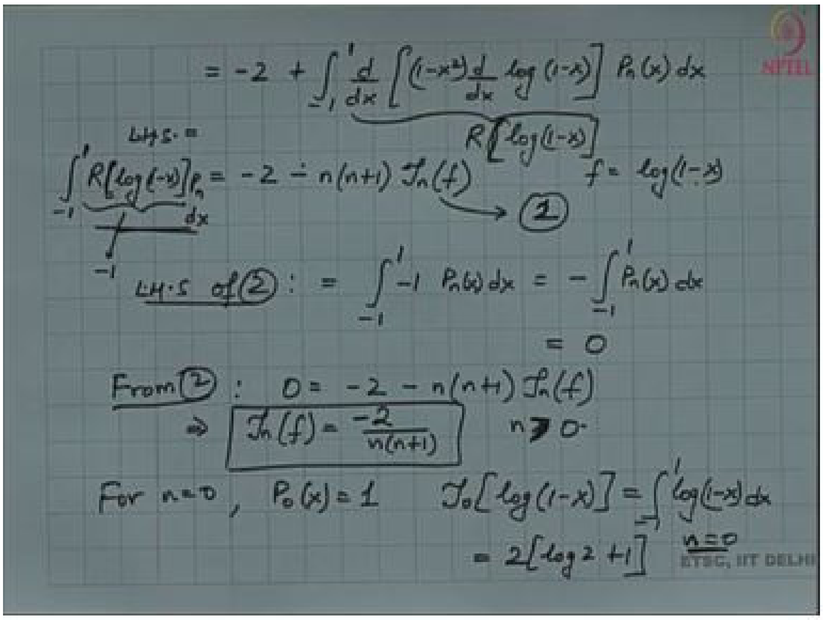
$$\Rightarrow \mathcal{J}_n\left[R\{\log(1-x)\}\right] = \int_{-1}^{1} R\left[\log(1-x)\right]P_n(x)dx$$

$$= \left(1-x^2\right)\frac{d}{dx}\log(1-x)P_n(x)\Big|_{-1}^{1} - \int_{-1}^{1} -(1+x){P_n}'(x)dx$$

$$\text{Note: } \ (1+x) = -\left(1-x^2\right)\frac{d}{dx}\log(1-x)$$

$$= -2 + \int_{-1}^{1}(1+x)P_n'(x)dx$$

$$= -2 - \left(1-x^2\right)\frac{d}{dx}\log(1-x)P_n(x)|_{-1}^{1} + I_0$$



$$= -2 + \int_{-1}^{1}\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\log(1-x)\right]P_n(x)dx$$

$$\int_{-1}^{1} R(\log(1-x))P_n = -2 - n(n+1)\mathcal{J}_n(f) \quad ...(2)$$

$$\text{LHS of (2)} = \int_{-1}^{1} -1 \quad P_n(x)dx = -\int_{-1}^{1} P_n(x)dx = 0$$

$$\text{From (2)} : 0 = -2 - n(n+1)\mathcal{J}_n(f)$$

$$\mathcal{J}_n(f) = \frac{-2}{n(n+1)} \qquad n > 0;$$

For $n = 0, P_0(x) = 1$,

$$\mathcal{J}_n[\log(1-x)] = \int_{-1}^{1}\log(1-x)dx$$

$$= 2[\log 2 + 1]$$