Integral Transforms and Their Applications

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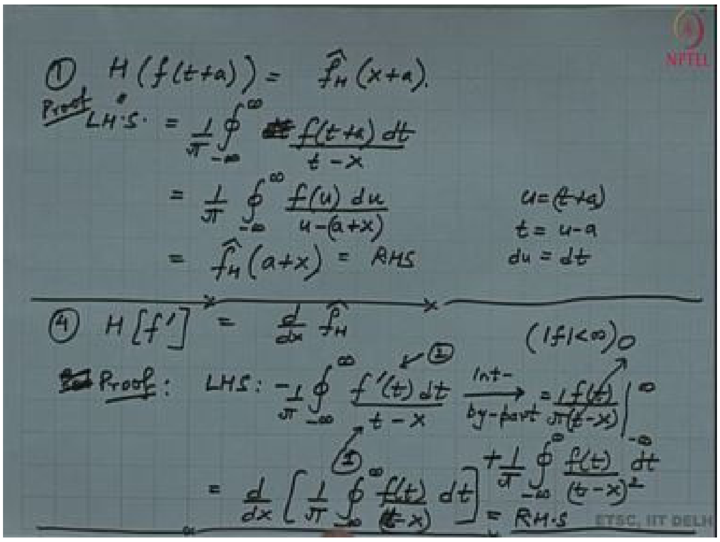
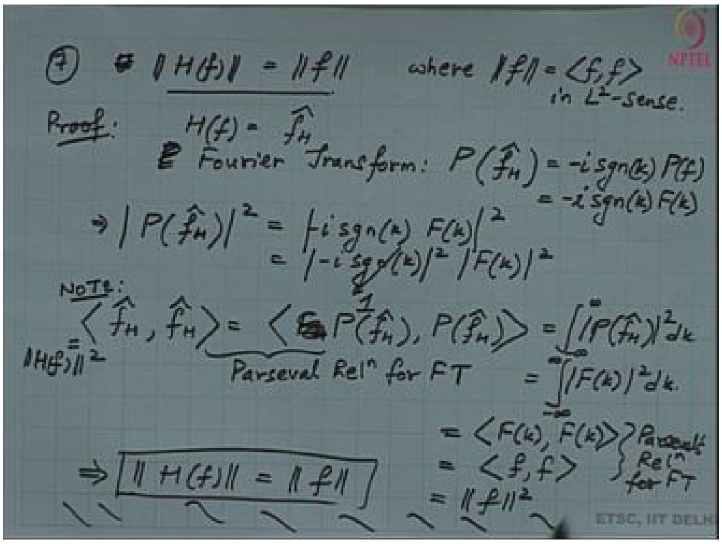
Lecture – 11

Introduction to Hilbert Transforms Part - 03



I need to show you the property 7, that I listed.



Proof 7:

$$\|H(f)\| = \|f\|$$

$$\text{where } ,||f|| =< f, f > \text{ in } \mathrm{L}^2 - \text{sense}$$

So, what I get is the following. I see that the Hilbert transform let me denote it by $H(f) = \hat{f}_H$. So, if I were to take the Fourier transform,

$$\Gamma\left(\hat{f}_H\right) = -i\,\mathrm{sgn}(k)\Gamma(f)$$
$$= -i\mathrm{sgn}(k)F(k)$$

if I were to take the absolute value of this function that is the Fourier transform and take a square.

$$\left|\Gamma\left(\hat{f}_H\right)\right|^2 = |i\;\mathrm{sgn}(k)F(k)|^2$$

$$= |-i\,\mathrm{sgn}(k)|^2|F(k)|^2$$

So, then so find it then using the fact that. So, notice that I have that the Hilbert transform if I take the inner product of the Hilbert transform. I get that this is also equal to the Fourier transform of:

Note:

$$\left\langle \hat{f}_H, \hat{f}_H \right\rangle = \left\langle \Gamma\left(\hat{f}_H\right), \Gamma\left(\hat{f}_H\right) \right\rangle = \int_{-\infty}^{\infty} |\Gamma\left(\hat{f}_H\right)|^2 dk$$

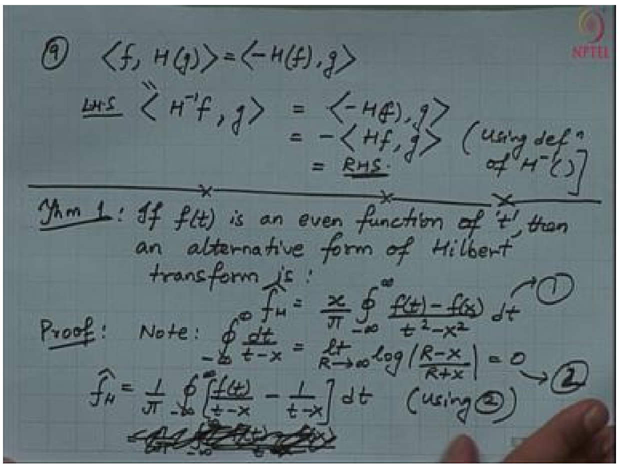
$$= \int_{-\infty}^{\infty} |F(k)|^2 dk$$

$$= \langle F(k), F(k) \rangle$$

$$= \langle f, f \rangle$$
$$= \|f\|^2$$

$$\|H(f)\| = ||f||$$

So, we have used the Parseval's relation to go from the Hilbert the inner product of the Hilbert function to the Fourier transform of the inner product of Hilbert function and going back from the Fourier transform of the function to the inner product of the function itself. And, this is nothing but the norm of the function square and to begin with we started with the norm of the Hilbert transform of the function square. Or, I get that the Hilbert transform of the function the norm of the Hilbert transform is the norm of the function itself and that is what I wanted to show to begin with, ok.



So, then there is this 9th relation. The property 9, let me just highlight the property 9 I had to show that :

Proof 9:

$$\langle f, H(g)\rangle = \langle -H(f), g\rangle$$

$$\text{LHS } \left\langle H^{-1}f, g\right\rangle = \langle -H(f), g\rangle$$

By using definition of the inverse transform of the Hilbert transform

$$= -\langle H(f), g\rangle$$

So, I leave the rest of the properties as an exercise for the students to complete, and see that the other properties are quite easily found using the definition of Hilbert transform.

So, then I have some results which I stated in the terms in the form of theorems.

Theorem 1:if I have a function $f(t)$ which is an even function. So, I am given a function $f(t)$ which is an even function of t, then an alternative form of Hilbert transform is as follows:

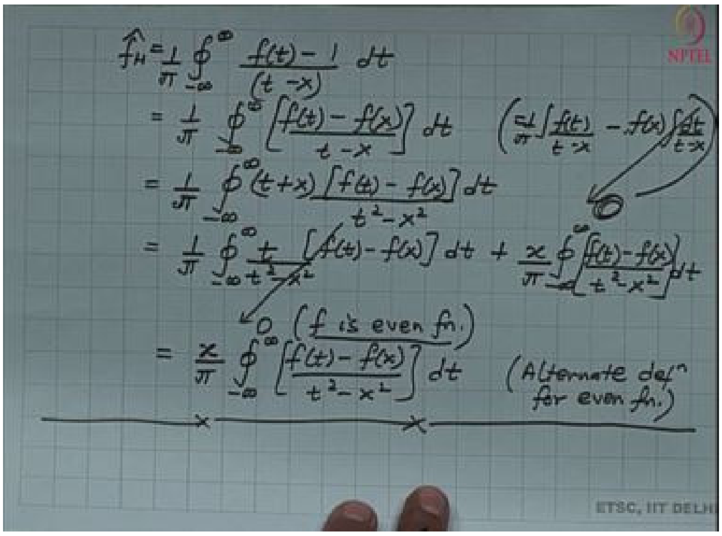
$$\hat{f}_H = \frac{x}{\pi}\oint_{-\infty}^{\infty}\frac{f(t)-f(x)}{t^2-x^2}dt \qquad ...(1)$$

Proof:

Note:

$$\oint_{-\infty}^{\infty}\frac{dt}{t-x} = \lim_{R\to\infty}\log\left(\frac{R-x}{R+x}\right) = 0 \qquad ...(2)$$

$$\hat{f}_H = \frac{1}{\pi}\oint_{-\infty}^{\infty}\left[\frac{f(t)}{t-x} - \frac{1}{t-x}\right]dt \qquad \text{using equation(2)}$$



$$\hat{f}_H = \frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{f(t)-1}{(t-x)}dt$$

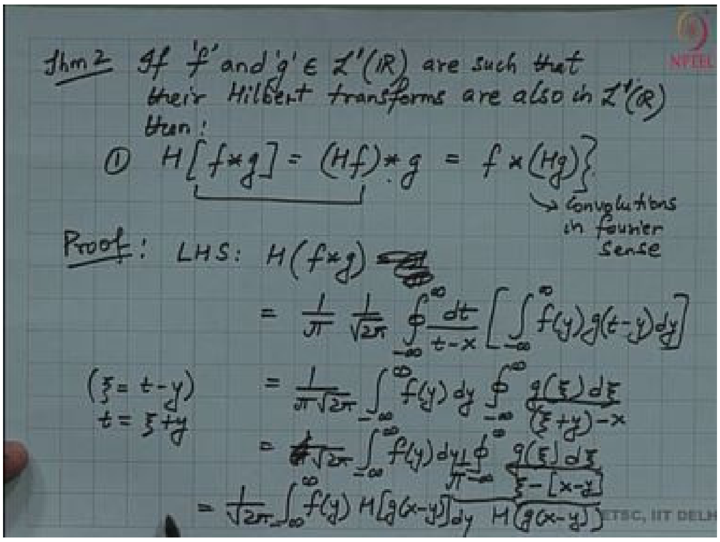
$$= \frac{1}{\pi}\oint_{-\infty}^{\infty}\left[\frac{f(t)-f(x)}{t-x}\right]dt$$

$$= \frac{1}{\pi} \oint_{-\infty}^{\infty} (t+x) \frac{[f(t) - f(x)]}{t^2 - x^2} dt$$

$$= \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{t}{t^2 - x^2} [f(t) - f(x)] \, dt + \frac{x}{\pi} \int_{-\infty}^{\infty} \left(\frac{f(t) - f(x)}{t^2 - x^2} \right) dt$$

$$= \frac{x}{\pi} \oint_{-\infty}^{\infty} \left[\frac{f(t) - f(x)}{t^2 - x^2} \right] dt$$

So, what we see is that if my function is even then I have an alternate definition of the Hilbert transform. So, this is an alternate definition for even function, ok. So, alternate definition for even function, ok. So, moving on I am going to start describing the convolution and the convolutions are straightforward in the case of Hilbert transforms. The convolutions are described using the Fourier transform. So, the convolutions are described in the Fourier sense.



Theorem 2:

If $'f'$ and $'g' \in \mathcal{L}'(R)$ are such that their Hilbert transforms are also in $\mathcal{L}'(R)$ then:

(1).

$$H[f * g] = (Hf) * g = f * (Hg)\}$$

Proof:

$$\text{LHS: } H(f * g)$$

$$= \frac{1}{\pi} \frac{1}{\sqrt{2\pi}} \oint_{-\infty}^{\infty} \frac{dt}{t - x} \left[\int_{-\infty}^{\infty} f(y) g(t-y) dy \right]$$

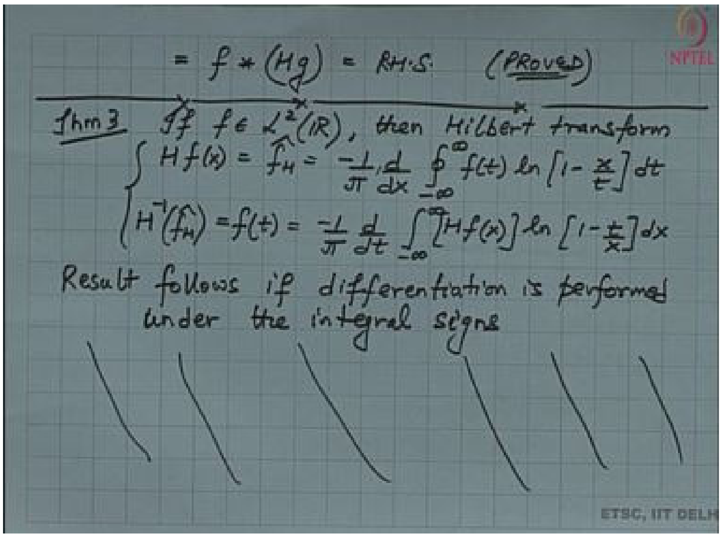
$$\text{Let,} \quad \begin{aligned} \xi &= t - y \\ t &= \xi + y \end{aligned}$$

$$= \frac{1}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) dy \oint_{-\infty}^{\infty} \frac{g(\xi) d\xi}{(\xi + y) - x}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)dy \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{g(\xi)d\xi}{\xi - [x-y]}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) H[g(x-y)]dy$$

$$= f * (Hg) = R \cdot H \cdot S$$



Theorem 3:

If $f \in \mathcal{L}^2(\mathbb{R})$, then Hildert transform :

$$Hf(x) = \hat{f}_H = \frac{-1}{\pi} \cdot \frac{d}{dx} \oint_{-\infty}^{\infty} f(t) \ln \left[1 - \frac{x}{t}\right] dt$$

$$H^{-1}\left(\hat{f}_H\right) = f(t) = \frac{-1}{\pi} \frac{d}{dt} \int_{-\infty}^{\infty} [Hf(x)] \ln \left[1 - \frac{t}{x}\right] dx$$

So, these are the two results. And let me just state that these two result follows if differentiation is performed inside the integral, differentiation is performed under the integral, so under suitable integral signs. So, under suitable conditions I can interchange my derivative operator with the integral operator, and when I do that, I can get this alternate definition of Hilbert transform and the inverse Hilbert transform, ok.