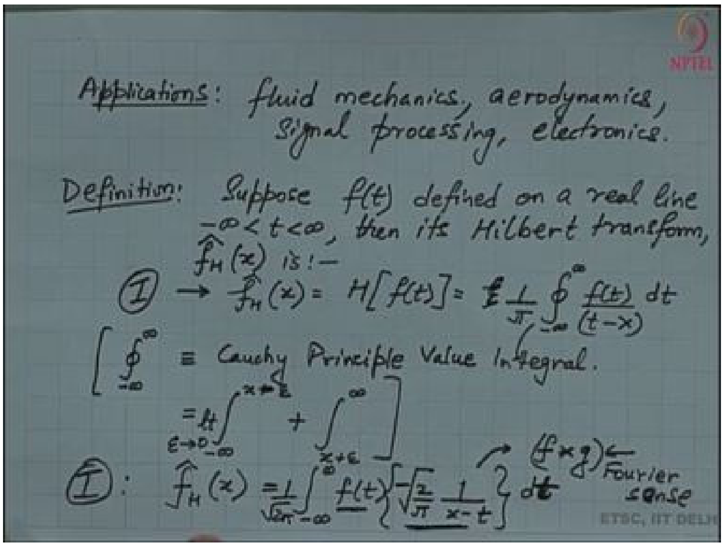
Integral Transforms and Their Applications

Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi

Lecture – 11
Introduction to Hilbert Transforms Part - 01

To recap in my previous lecture we have seen the applications, the properties and the definition as well for the Mellin transform. In today's lecture I am going to introduce yet another new transform known as the Hilbert transform. So, people specially in the electronics and communication background or in aerospace background they should be quite familiar with the utility and the importance of this transform. So, I am another fact is that Hilbert transforms are closely associated with the Fourier transforms. In particular we will see that Hilbert transforms are the product of the Fourier transforms in the convolution sense. So, without waiting much let us look at what are these transforms.



So, Hilbert transforms have lot of applications. So, before we move on to the definition I wanted to highlight the applications of Hilbert transforms which are not limited but definitely present in these areas. So, we will see lot of applications in the areas of fluid mechanics, in the areas of aerodynamics and signal processing as well as in electronics. So, let us look at what is the Hilbert transform of a function. So, I am going to assume a function a piecewise continuous function, let f of t we defined on a real line ok.

So, I am going to start with the function being real and I take my variable t, the argument of this function taking any value on the real axis, then its Hilbert transform

$$\hat{f}_H(x) = H[f(t)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{(t-x)} dt$$

Now notice that this is not this particular integral, let me just talk a little bit about this integral. So, this particular integral where I have specified the range of the integral that is this is a definite integral over the real axis.

So, in general this is not well in particular this is not a contour integral, this integral is also called as the Cauchy Principle Value Integral, So, in when we have to evaluate the Cauchy principle value integral we need to evade this singularity. So, this integral boils down to the following sum of two integrals.

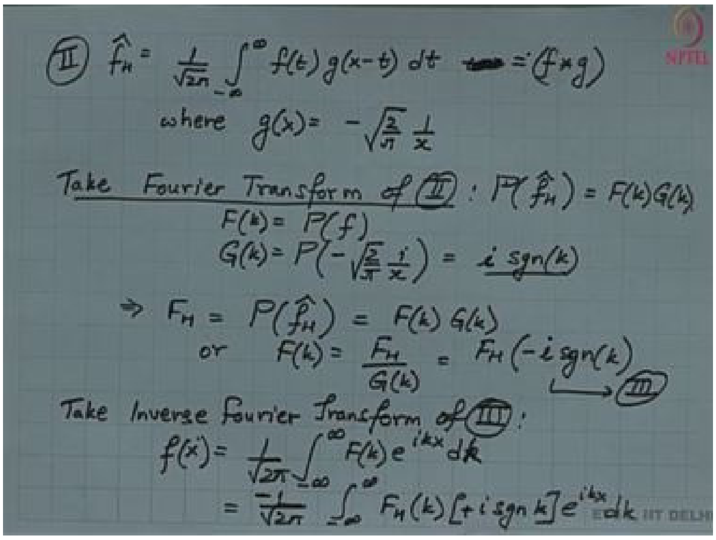
$$\begin{aligned} \oint_{-\infty}^{\infty} &= \text{ Cauchy Principle value Integral} \\ &= \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{x+\varepsilon} + \int_{x+\varepsilon}^{\infty} \right] \end{aligned}$$

So, and we take this limit epsilon going to 0 so that we are as close to the point of singularity as possible. So, again to repeat this particular integral is not a contour integral, but the principle value integral which is evaluated over the real axis in and in doing so, we are evading the point of singularity which is x. So, again to recap let us again look at the Hilbert. So, let me call this as my definition I, the definition of Hilbert transform. So, I, the definition in I is as follows: I see that this is also equal to,

$$\hat{f}_H(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left\{ -\sqrt{\frac{2}{\pi}} \frac{1}{x-t} \right\} dt$$

So, we see that this particular integral is written in the form of, in the form of the product of two function. So, this or this is in the form of the convolution of f star with g in the Fourier sense. So, I have written the Hilbert transform as a convolution of two functions in the Fourier sense.

So, again to rewrite my Hilbert transform is the convolution of 2 function



$$\hat{f}_H = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt = (f * g) \qquad .(2)$$

So, then let us take let me call this as my expression 2, again the definition in the product sense in the in the convolution sense.

$$\text{where } g(x) = -\sqrt{\frac{2}{\pi}}\frac{1}{x}$$

So, we take a Fourier transform, we take a Fourier transform of (2) to see what happens ok. So, we say that so, we take a Fourier transform of the Hilbert transform of f and that is equal to the product of the Fourier transforms of the two function,

$$\Gamma\left(\hat{f}_H\right) = F(k)G(k)$$

$$F(k) = \Gamma(f)$$

$$G(k) = \Gamma\left(-\sqrt{\frac{2}{\pi}}\frac{1}{x}\right) = i\,\mathrm{sgn}(k)$$

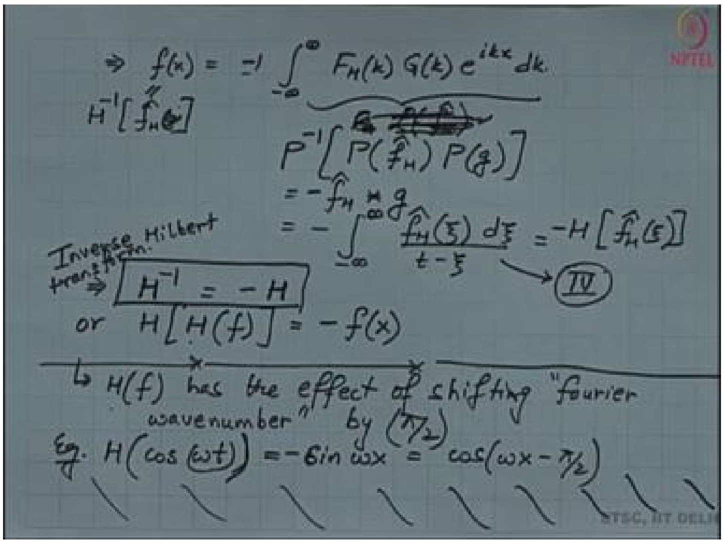
$$F_H = \Gamma\left(\hat{f}_H\right) = F(k)G(k)$$

$$\text{or} \quad F(k) = \frac{F_H}{G(k)} = F_H(-i\,\mathrm{sgn}(k))qquad(3)$$

So, now, let me call this expression as this equation as 3. So, I am going to take my inverse Fourier transform I am going to take my inverse Fourier transform of 3 to get the following.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$

$$= \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_H(k)[+isgn(k)]e^{ikx}dk$$



$$f(x) = -\int_{-\infty}^{\infty} F_H(k)G(k)e^{ikx}dk$$

$$
\begin{aligned}
&= -\hat{f}_H * g \\
&= -\int_{-\infty}^{\infty} \frac{f_H(\xi)}{t-\xi} = -H\left[\hat{f}_H(\xi)\right]
\end{aligned}
\qquad (4)
$$

So, what I have shown here is that the Hilbert inverse the inverse transform of a particular function is well be the minus sign is negative of the Hilbert transform or I see that the Hilbert transform of the Hilbert transform of the function f gives me negative of f ok. So, in this particular example the transform inverse of the transform is the same transform with a negative sign. So, the inverse transform is nothing but the same transform with a negative sign ok. So, so, that defines my Hilbert transform as well as the inverse. So, this is my definition of the inverse Hilbert transform or I can use this as my definition let me call this as (4). So, the inverse of the Hilbert transform is defined here by this expression on the right hand side which I call it (4).

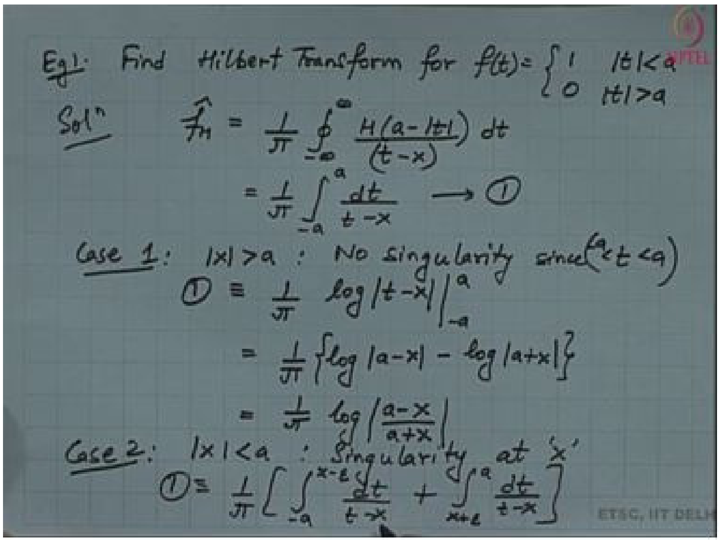
So, moving on so, before I move on there is just one more one more comment that I want to pass on. Notice that the Hilbert the effect of Hilbert transform is that two applications of Hilbert transform is shifting the function by negative one. So, if I were to talk in complex plane one two applications of Hilbert transform is shifting the angle of if suppose this function f was defined on the complex plane it is shifting the orientation of the function by 180 degrees.

So, -1 denotes a phase shift by 180 degrees which means the effect of Hilbert transform is to shift H of f has the effect of shifting the Fourier wave number Fourier wave number by $\frac{\pi}{2}$.

You see that the application of two Hilbert transforms shifts the phase by pi negative one or correspondingly the application of one Hilbert transform is going to phase shift by an angle $\frac{\pi}{2}$. We will see what this effect is very soon in some of our example. So well, so, just to give you an idea it means that if I were to use H of let us say example.

Example:

$$H(\cos(\omega t)) = -\sin \omega x = \cos(\omega x - \pi/2)$$



Example 1:Find the Hilbert transform for :

$$f(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$

Solution:

$$\hat{F}_H = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{H(a-|t|)}{(t-x)} dt$$

$$= \frac{1}{\pi} \int_{-a}^{a} \frac{dt}{t-x} \to 1$$

$$\text{case 1 :} \quad |x| > a : \text{ No singulanity since } (-a < t < a)$$

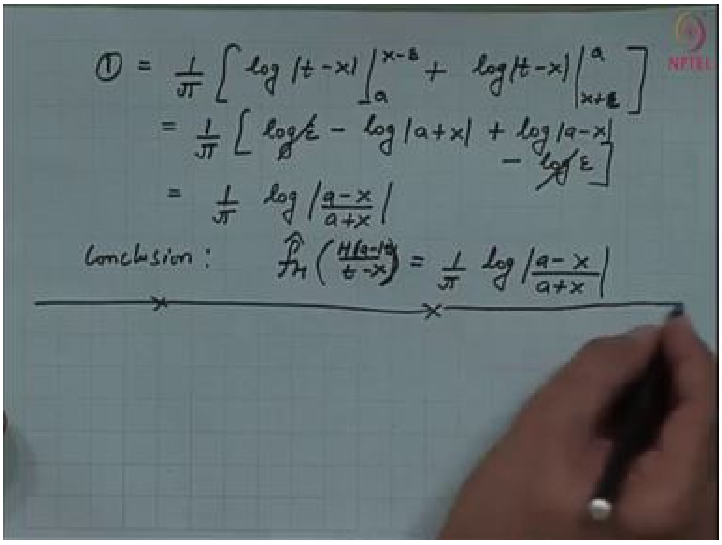
$$\text{equation(1)} = \frac{1}{\pi} \log|t-x| \Big|_{-a}^{a}$$

$$= \frac{1}{\pi} \int \log|a-x| - \log|a+x| \Big\}$$

$$= \frac{1}{\pi} \log\left|\frac{a-x}{a+x}\right|$$

$$\text{Case 2 :} \quad |x| < a : \text{ Singularity at } x$$

$$\text{equation(1)} = \frac{1}{\pi}\left[\int_{-a}^{x-\varepsilon} \frac{dt}{t-x} + \int_{x+\varepsilon}^{a} \frac{dt}{t-x}\right]$$



$$= \frac{1}{\pi}\left[\log|t-x|\right]_{a}^{x-\varepsilon} + \log|t-x|]_{x+\varepsilon}^{a}]$$

on solving,

$$= \frac{1}{\pi} \log\left|\frac{a-x}{a+x}\right|$$

$$\text{Conclusion:} \quad \hat{f}_H\left(\frac{H(a-|t|)^2}{t-x}\right) = \frac{1}{\pi}\log\left|\frac{a-x}{a+x}\right|$$

So, we have evaluated all the cases to find that the Hilbert transform comes out to be identical in whether there is a singularity in the integral or whether there is no singularity ok. So, let us move on.