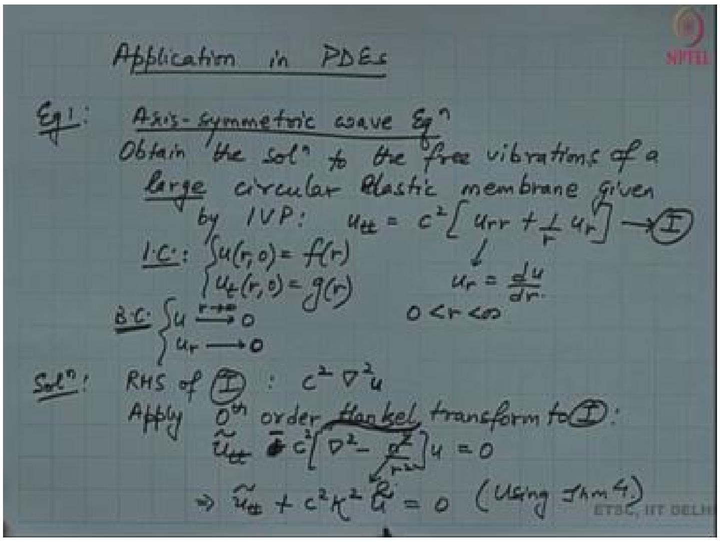
Integral Transforms and Their Applications

Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi

Lecture – 9
Introduction to Hankel Transforms- 03



Applications in PDEs:

So, applications; so, I am going to straight away dive into applications in PDEs ok. So, let us look at the solution to the axis symmetric wave equation ok. So, typically axis symmetric wave equation arises in problems related to vibrations let us say vibrations of some circular domains. Ex 1: Axis Symmetric Wave Equation:

So, let me just write down the statement of the problems. The this question says obtain the solution to the free vibration ok. So, when I say free vibration which means that the vibrations they die out as r the radius of the circular domain goes to infinity ok. So, the solution to the free vibrations of a large circular membrane circular well let me also add the word elastic because the membrane is vibrating ok.

So, elastic membrane given by the IVP:

$$u_{tt} = c^2 \left[u_{rr} + \frac{1}{r} u_r \right] \rightarrow (I)$$

I am given the initial condition, the initial condition is as follows;

$$u(r, 0) = f(r)$$
$$u_t(r, 0) = g(r)$$

Notice that we do not need the boundary condition in this case because we have the fact that u must go to must be bounded. So, u must go to 0 as r goes to ∞ which brings us to the fact that $0 < r < \infty$ because this is a large circular domain. So, we can assume that my radius of the domain is from 0 to ∞. So, that is my boundary condition and we can safely assume that the derivatives also vanish. So, those could be my boundary conditions ok. So, this is implied in this problem ok.

So let us look at the solution to the problem. As we see that there is no angular variation in this problem. So, the solution purely depends on the variable t and the variable r. So, there is no angular variation to the problem which means the natural choice of transform in this problem is the Hankel transform.

Solution:

Now, notice that the right hand side let us call this as my equation I, the RHS of I is:

$$= c^2 \nabla^2 u$$

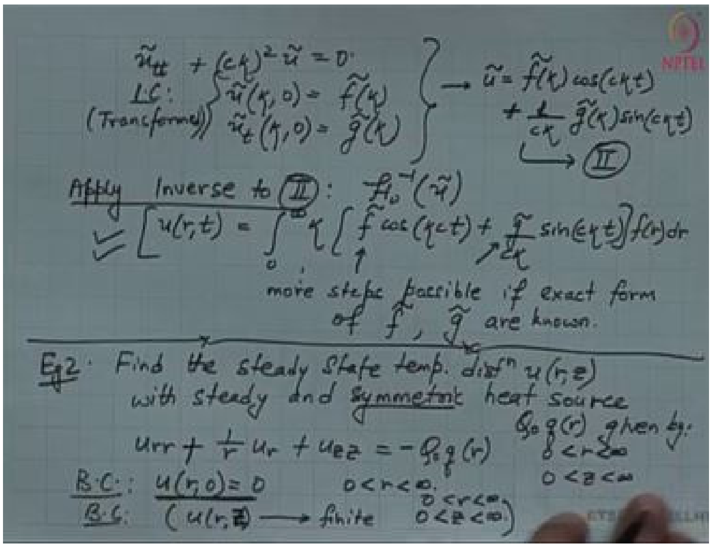
So, if I were to use my. So, let us now apply let us say 0th order Hankel transform 0th order Hankel transform ok because it becomes very easy when we apply the 0th order. So, I see that well this is not going to be changed the left hand side so transform to 1.

I get the following.

$$\tilde{u}_{tt} - c^2 \left[\nabla^2 - \frac{n^2}{r^2} \right] u = 0$$

$$\Rightarrow \tilde{u}_{tt} + c^2 k^2 \tilde{u} = 0 \qquad \text{using Theorem (4)}$$

So, then I have to solve this ODE this time a second order ODE.



$$\tilde{u}_{tt} + (ck)^2 \tilde{u} = 0$$
$$\text{I.C} : \tilde{u}(k, 0) = \tilde{f}(k)$$
$$(\text{ Transformed }) \quad \tilde{u}_t(k, 0) = \tilde{g}(k)$$

So, what I get is using my initial conditions, I am going to get the following answer to this problem, the answer I am. So, this can be easily integrated twice it will give a solution in terms of sine and cosine trigonometric functions and using these two initial conditions

$$\tilde{u} = \tilde{f}(k) \cos(ckt) + \frac{1}{ck} \tilde{g}(k) \sin(ckt) \qquad (2)$$

So, then the next stage is let us call this expression as my (2). So, if I apply inverse transform apply inverse to (2) that is:$\mathcal{H}_0^{-1}(\tilde{u})$

$$u(r,t) = \int_0^\infty k\left[f\cos(kct) + \frac{\tilde{g}}{ck}\sin(ckt)\right]f(r)dr$$

Notice that now up to this stage the problem has been solved, but after this stage we need more information about f tilde and g tilde to proceed with a complete evaluation of this integral right. So, I am going to leave this problem here.

So, more steps by saying that more steps possible if exact form of my functions the transformed functions $\tilde{f}$and$\tilde{g}$ are known ok. So, I can leave this expression here saying that this is my answer to the problem. Notice that this is just want to give you the highlight that notice this is a wave equation.

Example 2: So, then let us look at one more example. So, I have the followings question, find the steady state find the steady state temperature distribution u(r,z) with steady and symmetric heat source ok. So, I am going to solve my steady state diffusion equation. So, that will be a Laplace equation with a source. Let us say a source is given by the following form which is again independent of theta. So, it is a axis symmetric heat source ok. so I am given so find the steady state temperature distribution with this heat source given by the following equation.

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = -Q_0 q(r)$$

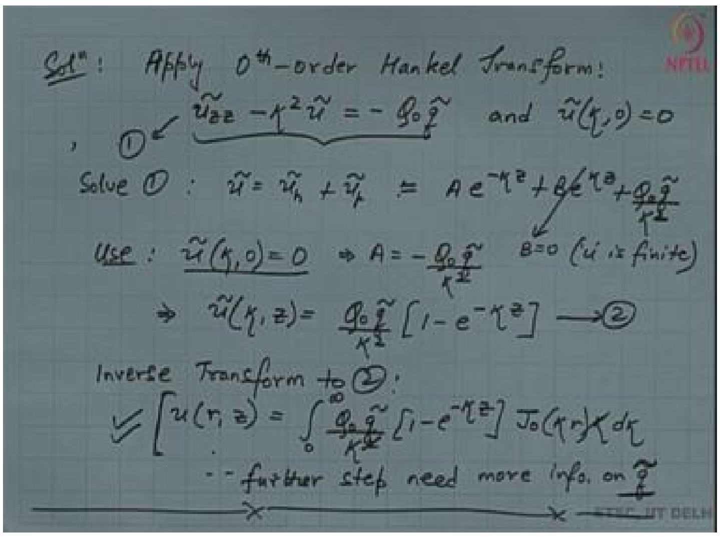
let us say it is a large cylinder and also z is from 0 to infinity.

Then I am also given an initial condition

$$B\cdot C: \quad u(r,0) = 0 \quad 0 < r < \infty$$

$$\text{B.C: } u(r,z) \to \text{ finite } 0 < z < \infty$$

So, that serves as my second boundary condition ok. So, let us look at the solution to this problem. Notice that this is already in the form of Laplacian this particular these two terms are the terms of the Laplacian, the radial component.



Solution:So, if I were to apply the Hankel transform. So, apply 0th order Hankel transform 0th order Hankel transform as follows:

$$\tilde{u}_{zz} - k^2\tilde{u} = -Q_0\tilde{q} \quad \text{and} \quad \tilde{u}(k,0) = 0 \qquad (1)$$

So, then I can see that this is if I were to solve this can be solved using the method of you know looking at this ODE in 2 ways. One is the homogeneous part. So, I can let me say that this is my 1. So, Solve 1:

$$\tilde{u} = \tilde{u}_h + \tilde{u}_p = Ae^{-kz} + B\, e^{kz} + \frac{Q_0\tilde{q}}{k^2}$$

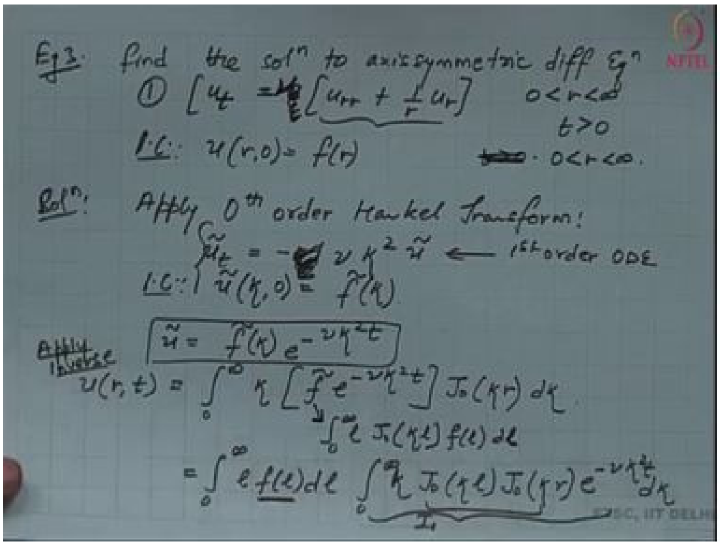
$$\text{use:} \quad \tilde{u}(k,0) = 0 \quad \Rightarrow A = -\frac{Q_0\tilde{q}}{k^2}$$

$$\Rightarrow \quad \tilde{u}(k,z) = \frac{Q_0\tilde{q}}{k^2}\left[1 - e^{-kz}\right] \rightarrow (2)$$

So, then finally, apply I am going to apply inverse transform, inverse transform to 2

$$u(r,z) = \int_0^\infty \frac{Q_0\tilde{q}}{k^2}\left[1 - e^{-kz}\right] J_0(kr)kdk$$

So, the further steps will need more information about q need more information on this transformed function q. So, I am to again leave this solution at this step saying that this is the solution to my axis symmetric diffusion equation and I can continue evaluating this integral provided I have an another analytic form of this transformed function $\tilde{q}$ ok.



So then I have another equation. So, let us look at the case;

Example (3): Find the solution to the axis symmetric diffusion equation

well. So, so this time it is again another problem, but this time with slightly different. So, this is going to be another situation of an axissymmetric diffusion, but this time not a steady state. So, I am going to solve axissymmetric diffusion equation this time the time dependent diffusion equation.

$$\left[u_t = u_{rr} + \frac{1}{r}u_{rr}\right] \qquad 0 < r < \infty, t > 0$$

And I am given the following initial conditions that

$$u(r, 0) = f(r)$$

Solution:If I apply 0th order Hankel transform I get the following.

$$\tilde{u}_t = -\nu k^2 \tilde{u} \leftarrow 1^{st} \text{ order ODE}$$

So, then my initial condition are transformed as well. So, I get

$$\tilde{u}(k, 0) = \tilde{f}(k)$$

So, I can I can see that this is a first order ODE first order ODE. And I can immediately find the solution to this problem given by

$$\tilde{u} = \tilde{f}(k)e^{-\nu k^2 t}$$

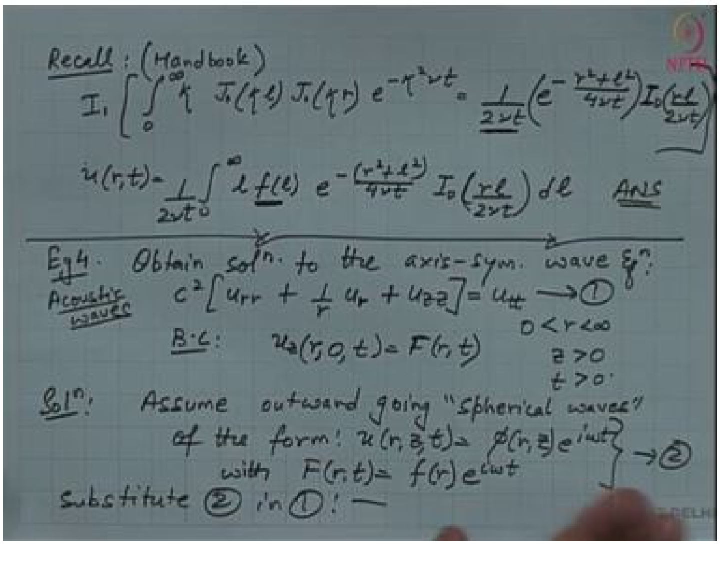
Notice that this ODE this solution satisfies the initial condition as well.

So, then the original solution in the original variable the physical variable will be after we apply inverse. So, once we apply inverse I get back the following

$$u(r, t) = \int_0^\infty k \left[\tilde{f} e^{-\nu k^2 t}\right] J_0(kr)dk$$

$$u(r, t) = \int_0^\infty ef(l)dl \int_0^\infty kJ_0(kl)J_0(kr)e^{-\nu k^2 t}dk$$

So, notice again we can only solve this further if we know the exact form of f ok. So, I am going to so, but the reason for introducing this second integral is we can still proceed a little bit further because we have a product of 2 Bessel function I can use one of my known results. So, let me just recall. So, let me say this is my integral I one.



So, if I were to recall one of the known results in Bessel function product. So, recall again go back to the handbook to see this expression the handbook that I have referred.

$$\left[I_1 \int_0^\infty kJ_0(kl)J_0(kr)e^{-\nu k^2 t} = \frac{1}{2\nu t}\left(e^{-\frac{r^2+l^2}{4\nu t}}\right) I_0\left(\frac{rl}{2\nu t}\right)\right]$$

$$u(r,t) = \frac{1}{2\nu t}\int_0^\infty \left[\ell f(\ell)e^{-\frac{\left(r^2+\ell^2\right)}{4\nu t}} I_0\left(\frac{r\ell}{2\nu t}\right) d\ell\right]$$

And this is the point where I will stop this example and move ahead because I cannot evaluate due to this unknown expression f of l ok. So, that is my answer to this problem ok.

So the solution the equation looks as follows. So, typically if we have sound waves say let us say sound waves arising from a particular source. The sound waves they are typically they emanate in radial fashion. So, they there is no angular preference for the emanate emanation of sound waves and hence they follow this axis symmetric wave equation. Example (4):obtain the solution this time again another axis symmetric problem obtain solution to the axis symmetric wave equation this time axis symmetric wave equation ok. Solution: Acoustic Wave:

$$c^2\left[u_{rr} + \frac{1}{r}u_r + u_{zz}\right] = u_{tt} \rightarrow (1) \quad 0 < r < \infty$$

The boundary condition:

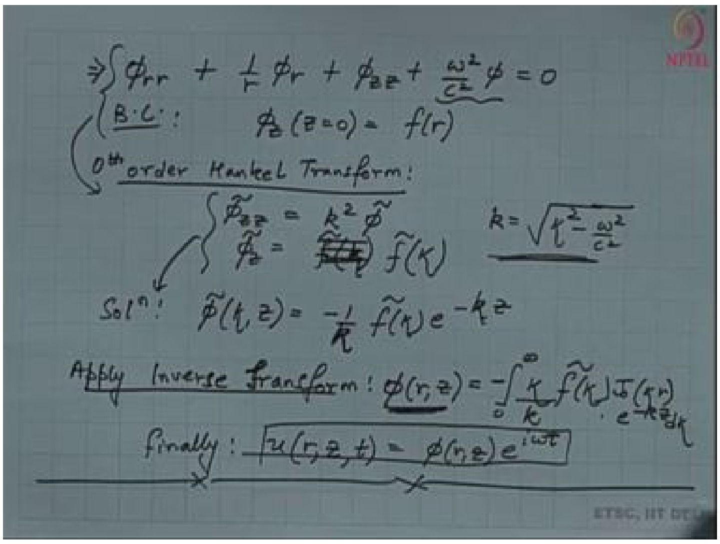
$$u_z(r,0,t) = F(r,t) \quad \begin{array}{l} 0 < r < \infty \\ z > 0 \\ t > 0 \end{array}$$

Solution: Assuming outwards going Spherical waves of the form:

$$u(r,z,t) = \phi(r,z)e^{i\omega t} \qquad ..(2)$$

$$\text{with, } F(r,t) = f(r)e^{i\omega t}$$

When I substitute 2 in 1 what I get is the following axis symmetric form of the wave equation ok.



$$\begin{aligned}&\phi_{rr} + \tfrac{1}{r}\phi_r + \phi_{zz} + \tfrac{\omega^2\phi}{c^2}\phi = 0\\ &\text{B.C :} \quad \phi(z=0) = f(r)\end{aligned}$$

So, then of course, if I apply my 0th order 0th order Hankel transform apply my 0th order Hankel transform I get that the equation looks as follows:

$$\begin{aligned}\tilde{\phi}_{zz} &= k^2\tilde{\phi} \quad ; k = \sqrt{k^2 - \tfrac{\omega^2}{c^2}}\\ \phi_z &= \tilde{f}(k)\end{aligned}$$

So then I can immediately find I have I know what is the solution to this problem this is a second order ODE the solution can be easily found by seeking the fact that the solution is:

Solution:

$$\tilde{\phi}(k,z) = -\frac{1}{k}\tilde{f}(k)e^{-kz}$$

And then of course, if I apply so solution to this and if I apply my inverse transform apply my inverse transform, I am going to get the following expression

$$\phi(r,z) = -\int_0^\infty \frac{k}{k}\tilde{f}(k)J_0(kr)\cdot e^{-kz}dk$$

finally my solution the original wave solution that we were seeking:

$$u(r,z,t) = \phi(r,z)e^{i\omega t}$$

So, that brings to the end of this problem. So, in the next lecture I am going to talk about another particular transform specially the so called Mellin transform. We will see that Mellin transform was is particularly useful in applications related to number theory. And in fact, it was shown by Riemann himself that Mellin transforms are quite useful in finding solution to the sum of infinite series.

So, thank you very much.