Integral Transforms and Their Applications

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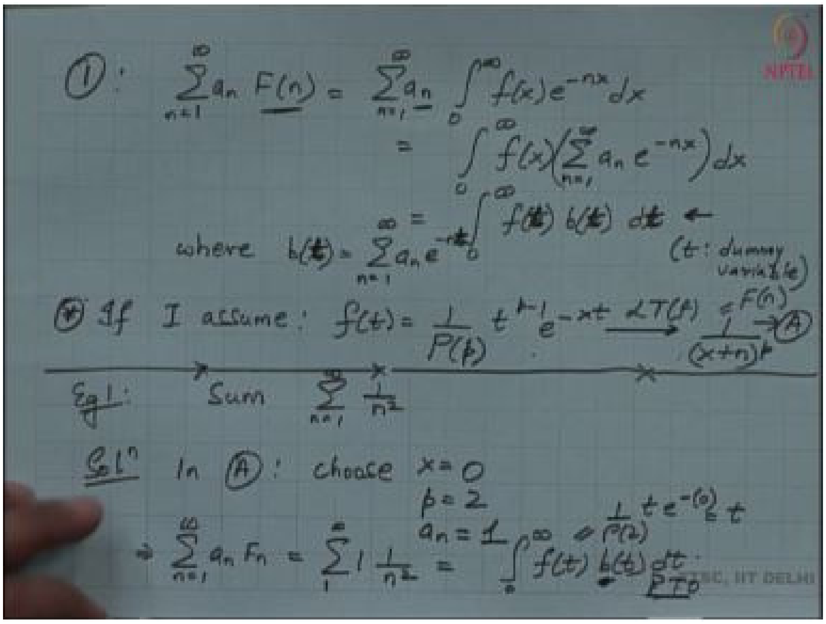
Lecture – 08

Applications of Fourier-Laplace Transforms Part 3



So sum let us look at this example:

Example 1:

$$\text{Sum1:} \sum_{n=1}^{3} \frac{1}{n^2}$$

So, I just want to remove any confusion well because t is nothing, but a dummy variable. So, instead of x I can replace it by some other variable t so then this is the coefficient b of t just to avoid confusion can confusion later on. So, then to sum this if I were to choose my in look at this expression here. So, I call this as A. So, in A in A if I choose my x to be my x to be 0 and my p to be 2 right. What I see is that that and also so this is my Laplace transform here right. And also I choose my coefficients:

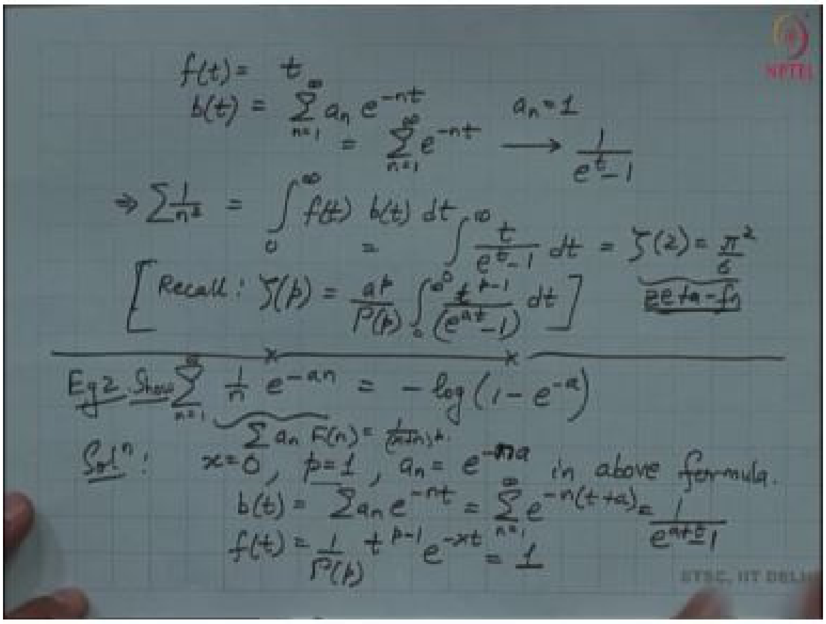
Solution:

$$\begin{aligned} \text{choose} \quad & x = 0 \\ & p = 2 \\ & a_n = 1 \end{aligned}$$

$$\sum_{n=1}^{\infty} a_n F_n = \sum_{1}^{\infty} 1 \frac{1}{n^2} = \int_0^{\infty} f(t) b(t) dt$$

and,

$$f(t) = \frac{1}{\rho(2)} \, \text{te}^{-(0)} = t$$



$$\text{We know,} \quad \begin{array}{l} f(t) = t \\ b(t) = \sum_{n=1}^{t} a_n e^{-nt} \end{array}$$

$$= \sum_{n=1}^{\infty} e^{-nt} \xrightarrow{a_n=1} \frac{1}{e^{t-1}}$$

$$\sum \frac{1}{n^2} = \int_0^{\infty} f(t) b(t) dt$$

$$= \int_0^{\infty} \frac{t}{e^t - 1} dt = \xi(2) = \frac{\pi^2}{6}$$

$$Recall : xi(p) = \frac{ap}{P(p)} \int_0^{\infty} \frac{t^{p-1}}{(e^{at} - 1)} dt$$

So, then let us look at some other examples I have the summation:

Example 2:

Show:

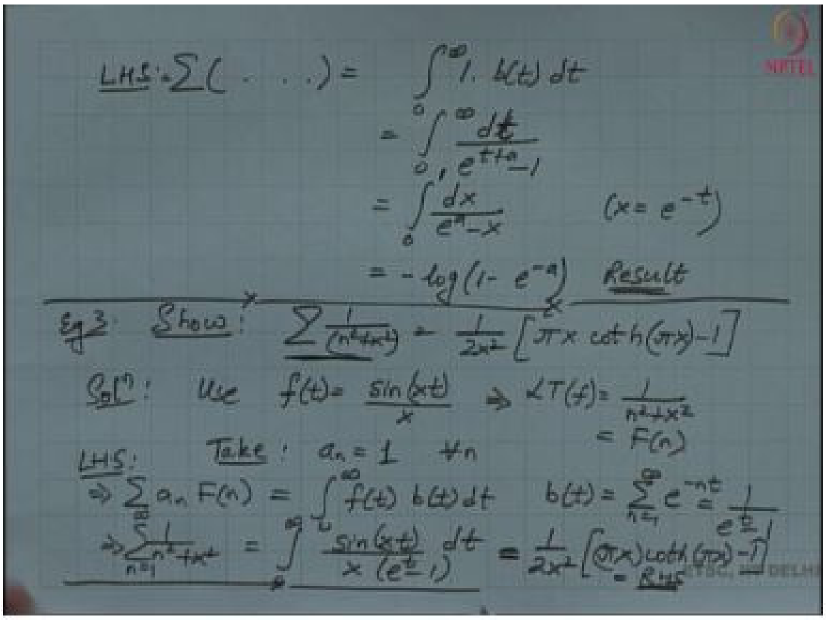
$$\sum_{n=1}^{\infty} \frac{1}{n} e^{-an} = -\log\left(1 - e^{-a}\right)$$

Solution:

$$x = 0, \quad p = 1, \quad a_n = e^{-na} \text{ in above formula.}$$

$$b(t) = \sum a_n e^{-nt} = \sum_{n=1}^{\infty} e^{-n(t+a)} = \frac{1}{e^{a+t}}$$

$$f(t) = \frac{1}{P(p)} t^{p-1} e^{-xt} = 1$$



$$= \int_0^\infty 1 \cdot b(t)dt$$

$$= \int_0^\infty \frac{dt}{e^{t+a} - 1}$$

$$= \int_0^1 \frac{dx}{e^a - x} \qquad \left(x = e^{-t}\right)$$

$$= -\log\left(1 - e^{-a}\right)$$

So, let us look at one more example.

Example 3:

Show:

$$\sum \frac{1}{(n^2 + x^2)} = \frac{1}{2x^2}[\pi x \cot h(\pi x) - 1]$$

Solution:

$$\text{use } f(t) = \frac{\sin(xt)}{x} \Rightarrow LF(f) = \frac{1}{n^2 + x^2} = F(n)$$

$$\text{Take : } a_n = 1, \forall n$$

$$\sum a_n F(n) = \int_0^\infty f(t)b(t)dt$$

$$b(t) = \sum_{i=1}^{\infty} e^{-nt} = \frac{1}{e^t - 1}$$

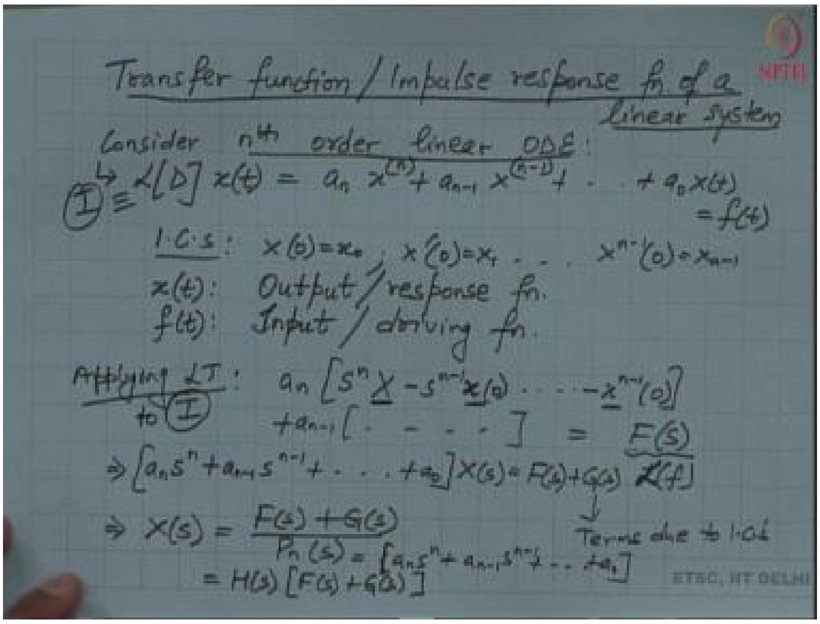
And we can easily integrate this let me write down the answer I leave some of the steps in between to the students to see that they arrived at this solution that is the RHS .

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2} = \int_0^\infty \frac{\sin(xt)}{x\left(e^t - 1\right)} dt$$

So, I get :

$$= \frac{1}{2x^2}[\pi x \cot h(\pi x) - 1]$$

Moving on, let me just quickly highlight one more aspect of this Fourier Laplace transform.



So, the transfer function and the impulse response function of a linear system. So consider, so consider an nth order linear ODE. So, let us look at this ODE and let us write it in compact form,

$$1 => L[D]x(t) = a_n x^{(n)} + a_{n-1}x^{n-1} + \cdots + a_0 x(t)$$

So, this is my linear ODE of the nth order. Now, with my following initial conditions:

$$\begin{aligned} &\text{ICs: } x(0) = x_0, x\prime(0) = x_1 \ . \ x^{n-1}(0) = x_{n-1} \\ &x(t) : \text{Output / response fn} \\ &f(t) : \text{Input / driving } f_n \ . \end{aligned}$$

So, typically you will see these impulse response, the concept of impulse response and transfer functions, when you have some signaling devices. So in a signaling devices, we try to provide an input. So, we let us say we provide an input signal to the device or we can call this device as some black box we provide an input signal and we want to figure out what is the output signal or the output or the response to this input signal to the device.

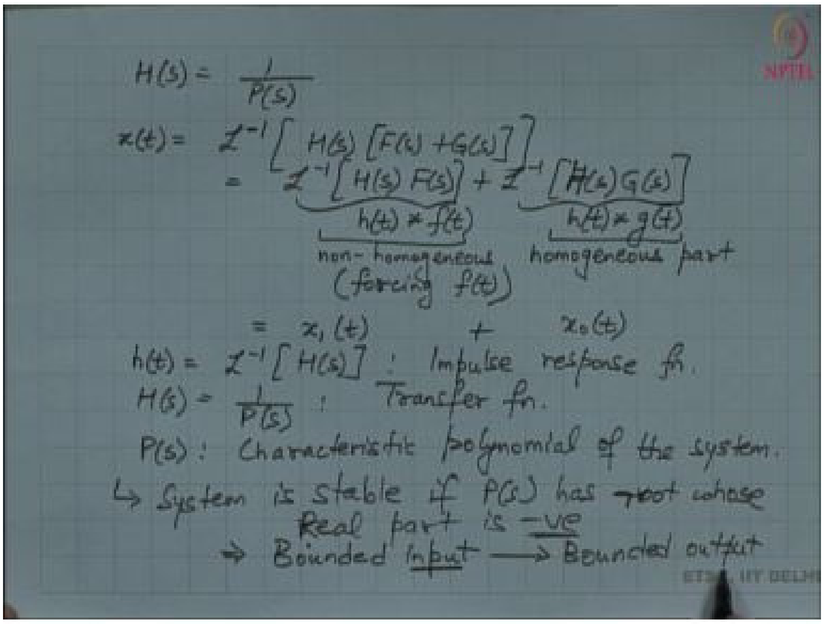
So, to find out we typically solve these nth order linear ODE's and try to figure out the quantities like these transfer functions, impulse response functions to find out the output signal of these devices. So, moving on so the variables x is the output or the response function f(t) is the input or the driving function. So, that is the notation that is used. So, we see that after applying Laplace transform:

Applying Laplace Transform to (1):

$$\begin{aligned} &a_n \left[s^n X - s^{n-1}x(0) \cdots - \cdots - x^{n-1}(0)\right] \\ &+a_{n-1}[\cdots]] = F(S) \end{aligned}$$

$$\left[a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0\right] \times (s) = F(s) + G(s)$$

$$x(s) = F(s) + G(s), \mathrm{P}_n(s) = \{a_n s^n + a_{n-1} s^{n-1}\} \ldots + a_1]$$
$$= H(s)[F(s) + G(s)]^n +$$



$$H(s) = \frac{1}{P(s)}$$
$$x(t) = L^{-1}[H(s)[F(s) + G(s)]]$$

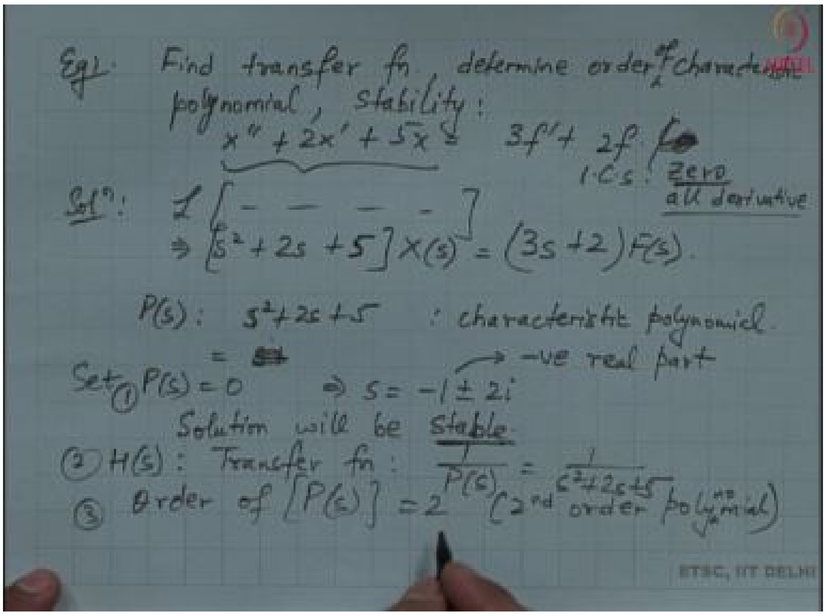
So, notice that this is going to come this is going to come from the homogeneous part. So, this is the homogeneous part of the ODE and this is coming out due to the non homogeneous part or due to the forcing due to the forcing f(t) on the right hand side. So, if we do not force our ODE on the right hand side then this part will vanish.

$$= L^{-1}[H(s)F(s)] + L^{-1}[H(s)G(s)]$$

$$= x_1(t) \quad + \quad x_0(t)$$

$h(t) = L^{-1}[H(s)]$: Impulse response fn
$H(s) = \frac{1}{P(s)}$: Transfer f_n .
$P(s)$: Characteristic polymomial of the system

Now, as long as P this we note that the system the system is stable if P has if P has negative has roots whose real part is negative whose real whose real part is negative we will see what I mean by that through some example. So, which means in that case we are going to get for a bounded for a bounded input I am going to get bounded output. So, which means if I force with a bounded function I am going to get a bounded solution to the ODE. So, let us look at some examples to see what is what do I mean by that. So, a very quick example is as follows:



So, find the transfer function. Find the transfer function and determine order of the characteristic polynomial of the characteristic polynomial, and discuss about the stability of the solution,So, my ODE is as follows:

Example 1:

Find transfer fn, detemine order of character
polynomial, statility:
$$x'' + 2x' + 5x' = 3f' + 2f$$

let us take the initial conditions to be 0 ,So, all so all derivatives are zeros. Solution:

$$\left(s^2 + 2s + 5\right) x(s) = (3s + 2) f(s)$$

$$P(s): \quad s^2 + 2s + 5 \quad \therefore \text{ characterstic polymonial.}$$

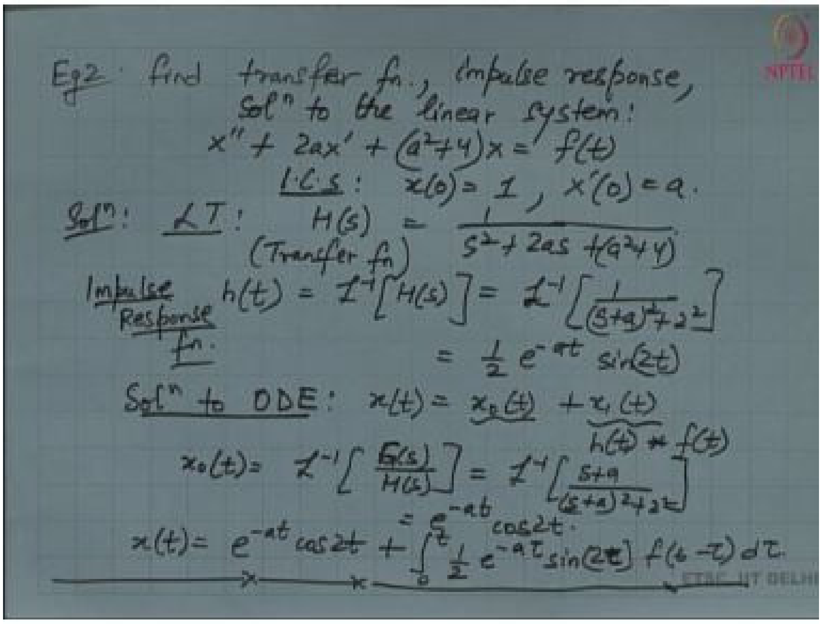
$$P(s) = 0 \quad \Rightarrow s = -1 \pm 2i$$

$$H(s): \text{ Transfer } f_n: \frac{1}{P(s)} = \frac{1}{s^2 + 2s + 5}$$

$$\text{order of}[P(s)] = 2$$

So, this is the second order polynomial second order polynomial. Note that the order the order is going to describe how many roots of P of s are there.

So, then let us now look at one more example, which will be the final example in today's discussion.



So, I have find the in the transfer function the transfer function, the impulse response the impulse response function and the solution to the linear system. And the system is given by the following:

Example 2:

Find transfer fn,impulse response and solution to the linear system:

$$x'' + 2ax' + (a^2+4)\,x = f(t)$$
$$(\text{ICS: } x(0) = 1, x'(0) = a$$

Solution:

Laplace Transform:

$$H(s) = \frac{1}{s^2 + 2as + (a^2+4)}, (transferfn)$$

So, I see that when I take the Laplace transform I immediately get that my transfer function.

Impulse response function:

$$h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{(s+a)^2 + 2^2}\right]$$

$$= \frac{1}{2}e^{-\alpha t}\sin(2t) \qquad (1)$$

So, then the solution once we know the impulse response function the solution to the ODE is quite straightforward. The solution to the ODE is:

Solution to ODE:

$$x(t) = x_0(t) + x_1(t) \qquad (2)$$

$$x_0(t) = L^{-1}\left[\frac{G(s)}{H(s)}\right] = L^{-1}\left[\frac{s+a}{(s+a)^2 + 2^2}\right]$$

$$= e^{-at} \cos 2t$$
$$x(t) = e^{-at} \cos 2t + \int_0^t \frac{1}{2} e^{-a\tau} \sin(2\tau) f(t-\tau) d\tau$$

So, so that completes that completes almost all my discussion on Laplace transform in this lecture. So, in the next lecture I am going to describe a new transform namely the Hankel transform. We will see that Hankel transforms especially useful when we have to solve problems having some sort of special symmetry. For example, axisymmetric problems problems with cylindrical symmetry. So, Hankel transforms are quite useful for specific class of problems. So, thank you very much for listening I will see you in the next lecture.

Thank you.