Integral Transforms and Their Applications

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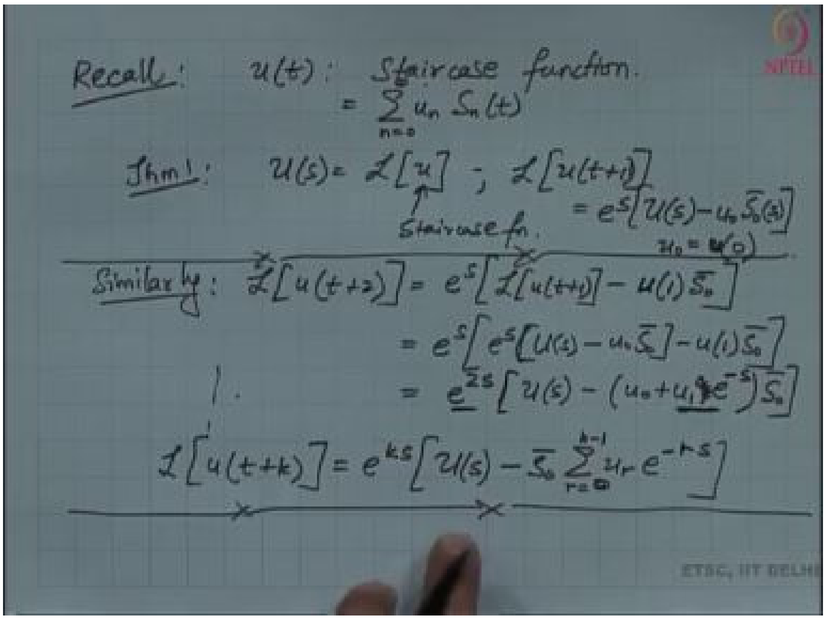
Lecture – 08
Applications of Fourier-Laplace Transforms Part 1

So, in the last lecture we saw how to use Laplace transform and say in the towards the end of the last lecture we saw, how to evaluate certain difference of functions and we were starting to solve certain difference equations and differential difference equations. So, today in this lecture we are going to continue and solve some examples in this category, and also the next half of this lecture will be all about using both Laplace and Fourier transform to solve problems in pd's and od's.



So, moving on let us now recall:

$$u(t): \text{ Staircase function}$$
$$= \sum_{n=0}^{\infty} u_n S_n(t)$$

So, this was a staircase function that we described. I saw that in theorem 1 I had a useful result:

$$u(s) = L[u]; L[u(t+1)] = e^s \left[u(s) - u_0 S(s)\right] u_0 = u(0)$$

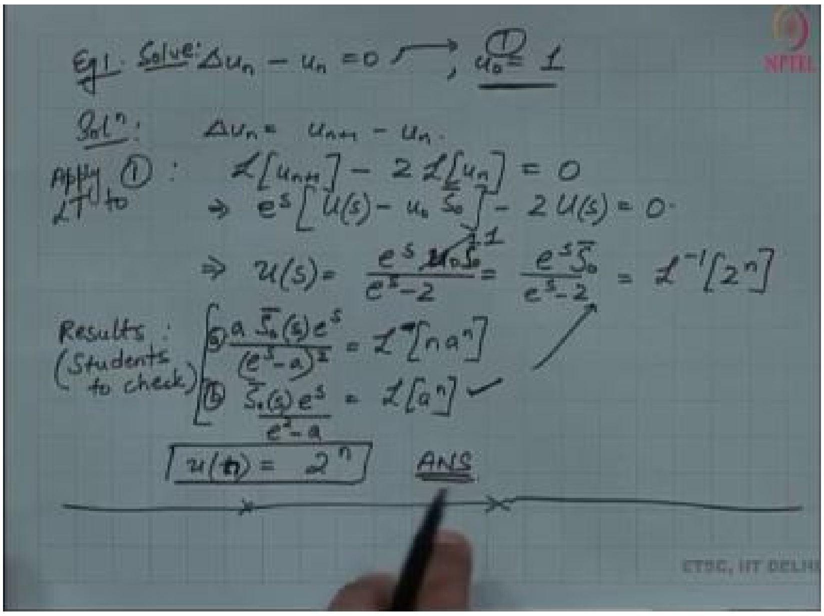
So, similarly with this result was already proved, we can look at similar results which will be useful to solve difference equation. So, we can for example: Example 1:

$$L[u(t+2)] = e^s \left[L[u(t+1)] - u(1)\overline{S_0}\right]$$

$$= e^{2s} \left[u(s) - \left(u_0 + u_1\right) e^{-s}\right] \overline{s_0}$$

$$L[u(t+k)] = e^{ks}\left[u(s) - \overline{S_0}\sum_{r=0}^{k-1} u_r e^{-ts}\right]$$

So, let us now look at some examples in this category, and how to solve difference equations.



So, look at this for example: Solve:

$$\Delta u_n - u_n = 0 \qquad\qquad 1$$

$$and, u_0 = 1$$

Solution:

$$\Delta u_n = u_{n+1} - u_n$$

So, when I apply Laplace transform to 1 I get that:

$$L\left[u_{n+1}\right] - 2L[u_n] = 0$$

$$e^s\left[u(s) - u_0\overline{S_0}\right] - 2u(s) = 0$$

So, we see that using this one of the 2 results namely thus the second one we get that well something is missing in this expression:

$$u(s) = \frac{e^s u_0 S_0}{e^s - 2} = \frac{e^s\overline{S}_0}{e^s - 2} = L^{-1}\left[2^n\right]$$

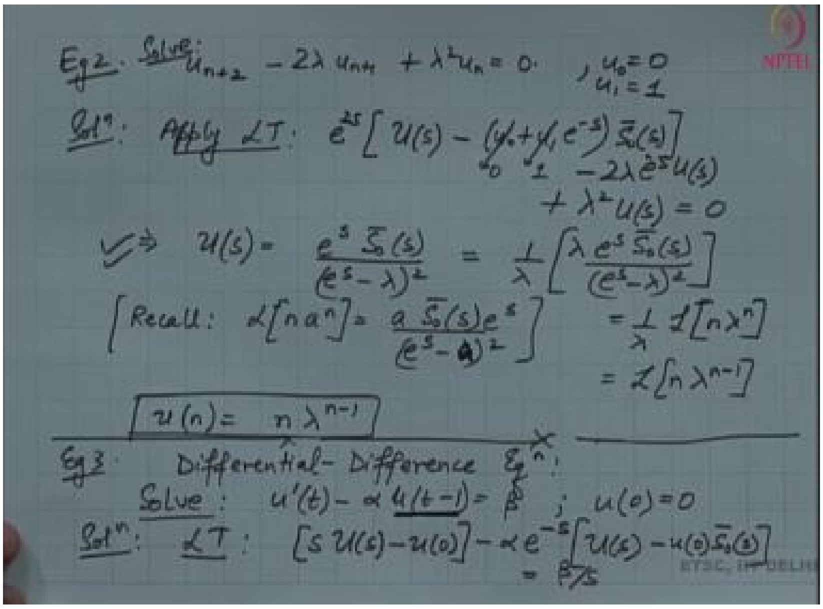
$$= \frac{a\overline{S_0}(s)e^s}{(e^s - a)^2} = L[na^n]\Big]$$

$$= \frac{\overline{S_0}(s)e^s}{e^2 - a} = L\left[a^n\right]$$

So,

$$u(n) = 2^n$$

So, I get my answer to this problem. So, moving ahead let us look at few more examples.



Example 2: Solve:

$$u_{n+2} - 2\lambda u_{n+1} + \lambda^2 u_n = 0, \quad , \begin{array}{l} u_0 = 0 \\ u_1 = 1 \end{array}$$

Now again I apply Laplace transform and I see that the first expression :

Applying Laplace transform:

$$\begin{array}{r} e^{2s}\left[u(s) - \left(u_0 + u_1 e^{-s}\right)\overline{S}_0(s)\right] \\ -2\lambda e^s u(s) + \lambda^2 u(s) = 0 \end{array}$$

So, simplifying all these expressions here, I am going to get the following :

$$u(s) = \frac{e^s \overline{S_0}(s)}{(e^s - 1)^2} = \frac{1}{\lambda}\left[\frac{\lambda e^s \overline{S_0}(s)}{(e^s - \lambda)^2}\right]$$

Recall,

$$L\left[na^n\right] = \frac{a\overline{S_0}(s)e^s}{(e^s - a)^2}$$

So, I get that my solution :

$$\begin{array}{l} = \frac{1}{\lambda} L\left[n\lambda^n\right] \\ = L\left[n\lambda^{n-1}\right] \end{array}$$

So, this is how we typically try to solve the difference equation using this staircase function definition.

So, then let us look at case of this is the case of a differential equation now. So, a differential:

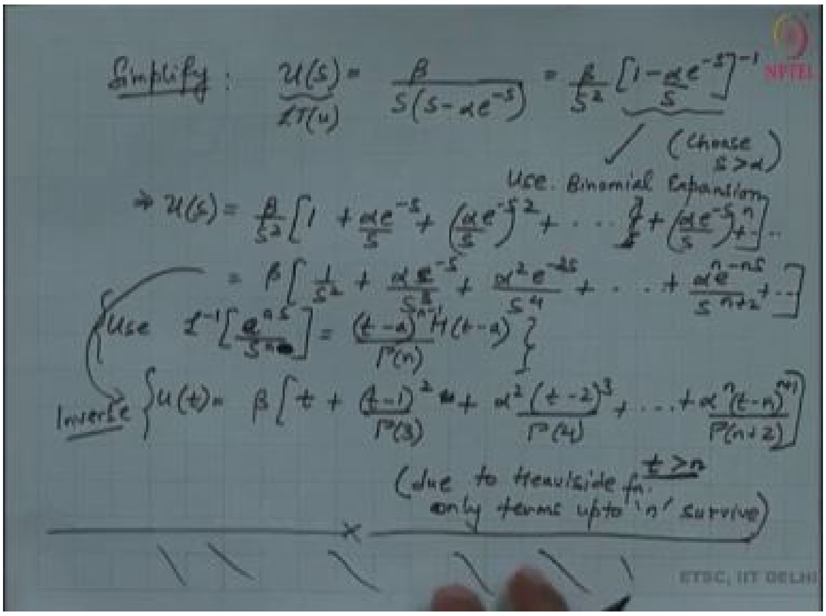
Example 3: Solve:

$$u'(t) - \alpha u(t-1) = \beta, u(0) = 0$$

So, again I apply Laplace transform, I get that the Laplace transform of the derivative is going to be,So, this we have to use the certain backward difference, I get that : Solution:

$$LaplaceTransform : [su(s) - u(0)] - \alpha e^{-s} \left(u(s) - u(0)\overline{s}_0(s)\right] = \frac{\beta}{s}$$

So, this is the difference this quantity is the difference with the difference taken backwards and here is a quantity which is the differential of the solution .



So, then after simplifying after simplifying, I get that:
Simplify:

$$L(u) = u(s) = \frac{\beta}{s\left(s - \alpha e^{-s}\right)} = \frac{\beta}{s^2}\left[1 - \frac{\alpha}{s}e^{-s}\right]^{-1}$$

Choose $s > \alpha$, I can expand this using binomial expansion,So, I use binomial expansion to come to this following series.:

$$u(s) = \frac{\beta}{s^2}\left[1 + \frac{\alpha e^{-s}}{s} + (\frac{\alpha e^{-s}}{s})^2 + \cdots + \left(\frac{\alpha e^{-s}}{s}\right)^n + \cdots\right]$$

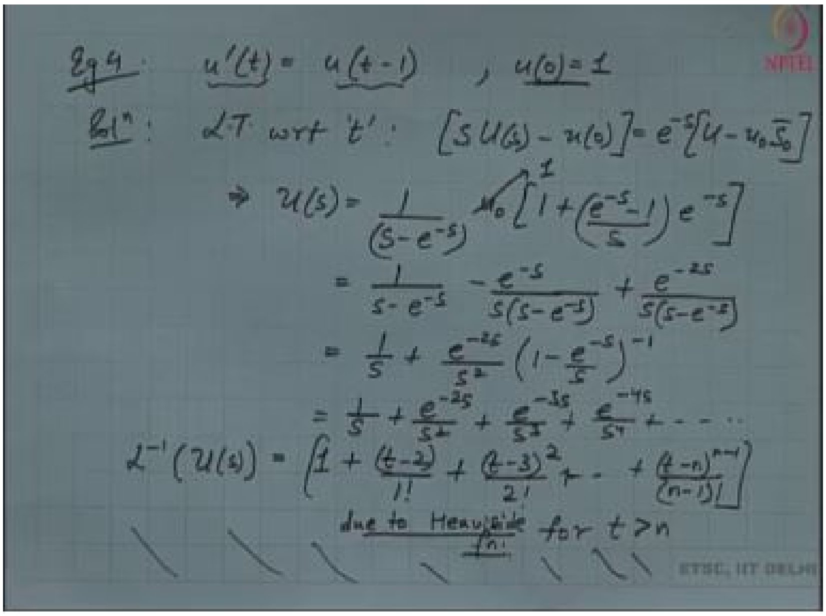
$$= \beta\left[\frac{1}{s^2} + \frac{\alpha e^{-s}}{s^3} + \frac{\alpha^2 e^{-\alpha s}}{s^4} + \cdots + \frac{\alpha^2 e^{-ns}}{s^{n+2}} + \cdots\right]$$

Now, using the fact we are going to use the fact that the Laplace transform of the Heaviside function is this exponential. So, or the inverse of this quantity the inverse of this quantity is the following expressions,So, I take the inverse and arrive at the result that this:

$$u(t) = \beta\left[t + \frac{(t-1)^2}{P(3)} + \alpha^2\frac{(t-2)^3}{P(4)} + \ldots + \frac{\alpha^n(t-n)^{n+1}}{P(n+2)}\right]$$

And the series has to end provided that I specify what time are we calculating, We see that for this time only these terms will survive the other terms in the series they will not survive. So, I have for u the solution to this differential difference equation are the some of these finite number of terms.

So, this happens due to the Heaviside function. So, only terms up to n survive the rest of the terms they are all 0. So, moving on let us look at one more example.



So, I have one more case where I am solving this difference differential difference equation: Example 4:

$$u'(t) = u(t-1) \quad , \quad u(0) = 1$$

So, again I take the Laplace transform with respect to t and I reduce the problem to the following:

Solution:

$$LT \text{ w.r.t } t' : \quad [su(s) - u(0)] = e^{-s} \left\{ \left(u - u_0 \overline{S_0} \right) \right.$$

$$u(s) = \frac{1}{(s - e^{-s})} u_0 \left[1^1 + \left(\frac{e^{-s} - 1}{s} \right) e^{-s} \right]$$

So, I get now the expression :

$$= \frac{1}{s - e^{-s}} - \frac{e^{-s}}{s\,(s - e^{-s})} + \frac{e^{-2s}}{s\,(s - e^{-s})}$$

$$= \frac{1}{s} + \frac{e^{-2s}}{s^2} \left(1 - \frac{e^{-s}}{s} \right)^{-1}$$

So, then after the simplification I am going to get that :

$$= \frac{1}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2} + \frac{e^{-4s}}{s^4} + \dots$$

$$L^{-1}(u(s)) = \left[1 + \frac{t-2}{1!} + \frac{(t-3)^2}{2!} + \dots + \frac{(t-n)^{n-1}}{(n-1)!} \right], t > n$$

And, only the n terms survive for $t > n$ as again because due to the Heaviside function . So, let us move on let us look at some more examples which are topic specific.