Integral Transforms and Their Applications
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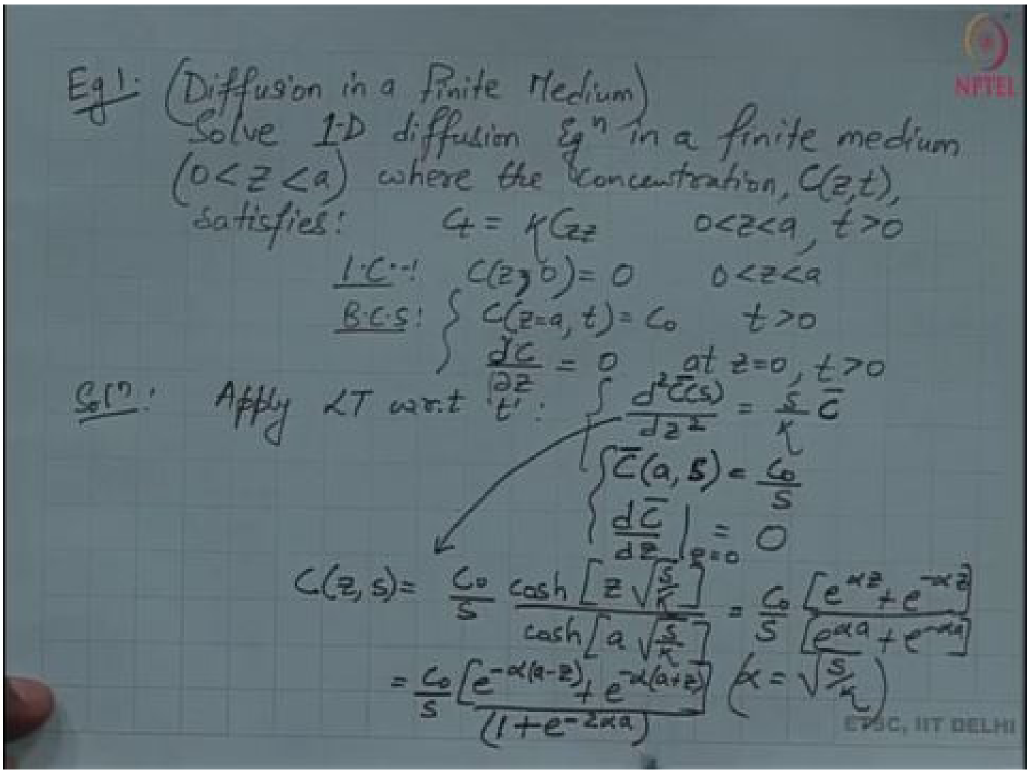
Lecture – 07
Applications of Laplace Transforms (Continued) Part 1

We will continue discussing some useful applications of Laplace transform specially related to problems in fluid mechanics, in problems to in applied sciences and engineering and I am going to use some of the useful properties that we have derived or proved specifically the Tauberian theorems and the heavy side expansion results in solving these problems. So, to begin with let me just give you an example:



So, this is the case of diffusion in a finite medium. So, the question says solve 1-D. So, solve 1-D diffusion equation in a finite medium. Let us say I define the medium to be: $0 < z < a$, where the concentration $C(z,t)$ satisfies the following PDE with the initial and the boundary conditions:

$$C_t = kC_{zz} \quad 0 < z < a, t > 0$$

So, the PDE to be solved is this diffusion equation, where my domain is defined on the interval time t is positive and my initial conditions are given as follows:

$$c(z,0) = 0 \quad 0 < z < a$$

and then two boundary conditions are:

$$C\left(z = a, t\right) = c_0 \quad , t > 0$$

$$\frac{\partial C}{\partial Z} = 0, \qquad \text{at } z = 0, t > 0$$

The concentration is prescribed at one end; the 1st condition is a Dirichlet condition and the second condition this one is a Neumann condition. So, then the solution methodology is as usual we are going to apply a Laplace transform with respect to one of the variables let us say t.So, t being the value, which is positive and goes from 0 to infinity. So, it is natural to apply Laplace transform with respect to t. So, then applying that I get the following set of transformed equation,Second derivative is: :

$$\frac{d^2\left(\bar{c}(s)\right)}{dz^2} = \frac{s}{k}\bar{c}$$

I have already used my initial condition in evaluating this ODE in finding this ODE. Now then the second the boundary condition reduces to the following :

$$\overline{C}(a, s) = \frac{c_0}{s}$$

$$\left.\frac{d\overline{C}}{dZ}\right|_{z=0} = 0$$

Now, the solution to this ODE is satisfying these two conditions become:

$$C(z, s) = \frac{c_0}{s}\frac{\cosh[z\sqrt{\frac{s}{k}}]}{\cosh[a\sqrt{\frac{s}{k}}]}$$

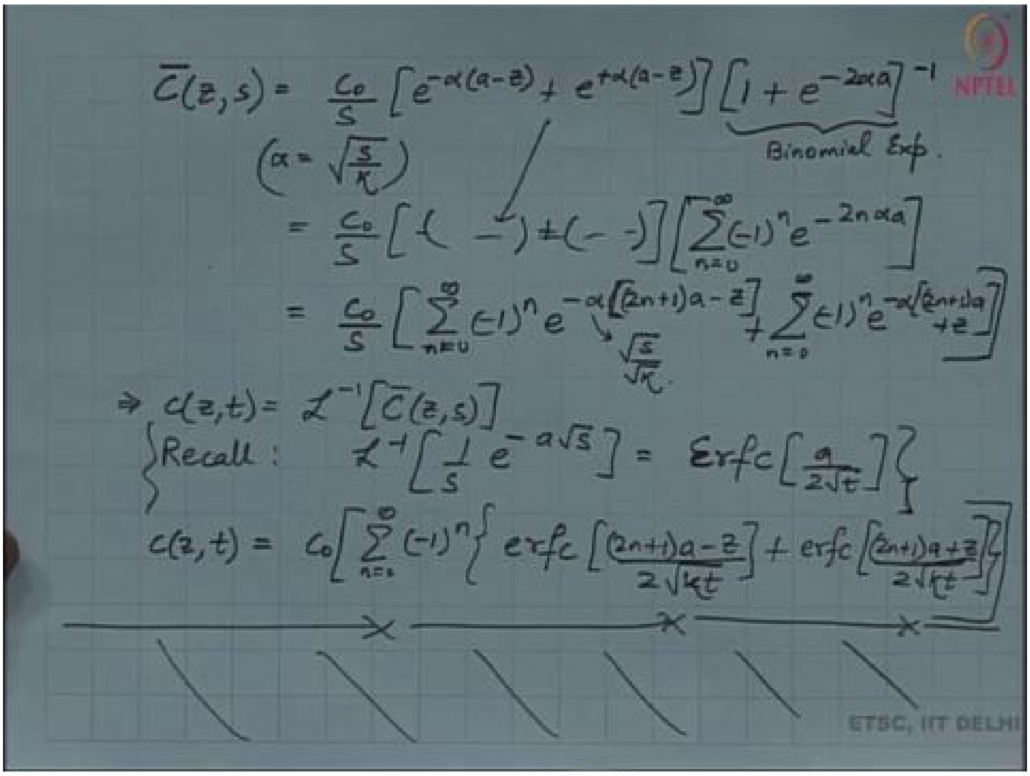
So at z equal to a this expression reduces to this constant this expression , $\frac{c_0}{s}$. So, then the next step is to well before we apply the inverse transform let us post process this expression a little bit.

$$\frac{c_0}{s}\left\{\begin{array}{l} e^{\alpha z} + e^{-\alpha z} \\ e^{\alpha a} + e^{-\alpha a} \end{array}\right.$$

=

$$\frac{c_0}{s}\left[\frac{e^{-\alpha(a-z)} + e^{-\alpha(a+z)}}{(1 + e^{-2\alpha a})}(\alpha = \sqrt{\frac{s}{k}})\right.$$

So, then I am going to for keep post processing this expression.



So, the expression :

$$\overline{C}(z,s) = \frac{c_0}{s}\left[e^{-\alpha(a-z)} + e^{+\alpha(a-z)}\right]\left[1 + e^{-2\alpha a}\right]^{-1}$$

,$(\alpha = \sqrt{\frac{s}{k}})$. So, I am going to open up this factor using binomial expansion.

$$\frac{c_0}{s}\left[e^{-\alpha(a-z)} + e^{+\alpha(a-z)}\right]\left[\sum_{n=1}^{\infty} -1^n e^{-2n\alpha a}\right]$$

When I use the binomial expansion I get a series expansion as follows:

$$\frac{c_0}{s}\left[\sum_{n=0}^{\infty} -1^n e^{-\alpha[(2n+1)a-z]} + \sum_{n=0}^{\infty} -1^n e^{-\alpha[(2n+1)a-z]}\right)$$

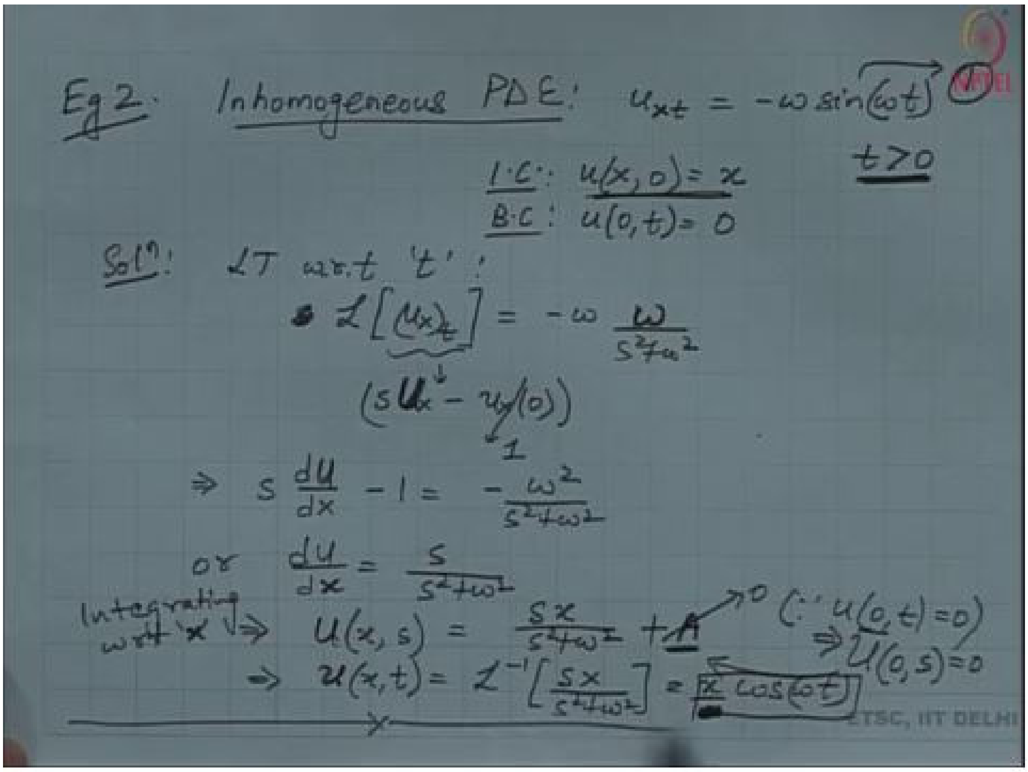
So, I get that I need to now evaluate the inverse of this expression. So

$$a(z,t) = \mathcal{L}^{-1}[\mathcal{C}\bar{(}z,s)]$$

And notice that these are my exponentials, $(\alpha = \sqrt{\frac{s}{k}})$. So, if we recall that we have a result, $L^{-1} - \left[\frac{1}{s}e^{-a\sqrt{s}}\right] = \varepsilon_r fc\left[\frac{a}{2\sqrt{t}}\right]$ So, if I were to use this result notice that this is a series, this transform is a series of those inverse transforms of error functions error function compliments. So, when I do the inverse transform I get the following result :

$$c(z,t) = c_0\left[\sum_{n=0}^{\infty}(-1)^n\left\{\text{erfc}\left[\frac{(2n+1)a-z}{2\sqrt{kt}}\right] + \text{erfc}\left[\frac{(2n+1)a+z]}{2\sqrt{kt}}\right]\right\}\right.$$

So, that completes the solution to this problem and we see that this solution quickly you know decays this is an error function compliment this quickly decays for few within rapidly within the first few terms of n unless we see that this function in the denominator of this error function is large enough. So, we see that if for long time this error function complement you know attains a constant value .



So, another problem that I want to highlight of which is the practical value is the case of inhomogeneous PDEs. So, let us consider a PDE of this form. So, I am going to solve solve a PDE where the derivatives are mixed derivatives:

$$u_{xt} = -\omega \sin(\omega t), t > 0 \qquad 1$$

So, my variable of interest is u and u is differentiated with respect to x and t. So, x and t are my independent variables So,initial condition ,

$$u(x, 0) = x$$

and boundary condition,

$$u(0, t) = 0$$

The solution can be again found by applying the Laplace transform let us say with respect to t. Notice that that t satisfies our criteria for this variable to be the choice for Laplace transform. So, then when we do that the first term in this let us say this is our expression 1; the first term in 1 gets is changed to following:

$$\mathcal{L}\left[(u_x)_t\right] = -\omega \frac{\omega}{s^2 + \omega^2}$$

So, we know

$$u_x(0) = 1$$

because of this following initial condition if I take a derivative I get a value equal to 1. So, then the Laplace this the equation reduces to the following :

$$s\frac{du}{dx} - 1 = -\frac{u^2}{s^2 + w^2}$$

$$\frac{du}{dx} = \frac{s}{s^2 + w^2}$$

Integrating,

$$U(x, s) = \frac{sx}{s^4u^2} + A$$

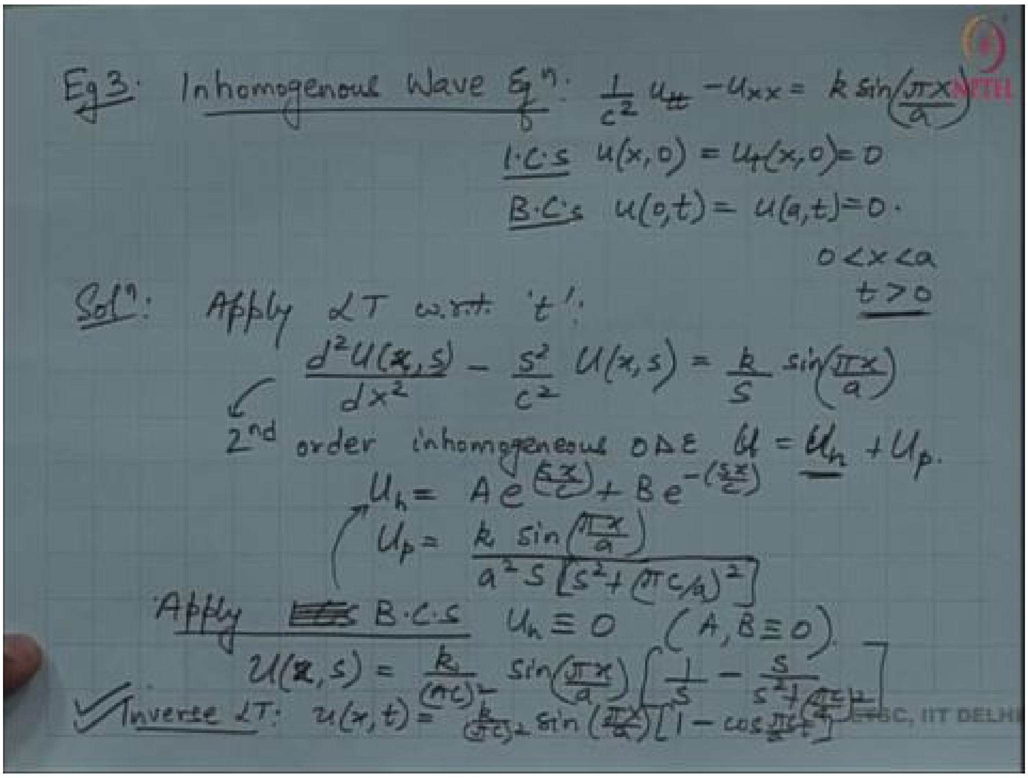
as u(0, t)=0

so u(0, s)=0

So,

$$u(x, t) = L^{-1}\left[\frac{Sx}{S^2 + w^2}\right]$$
$$= x\cos(\omega t)$$

and that is the answer to this PDE.



Moving on let us look at another case that is the case of inhomogeneous wave equation. So, the equation that is of our concern is:

$$\frac{1}{c^2}u_{tt} = -u_{xx} = k\sin\left(\frac{\pi x}{a}\right)$$

I am given the following initial and boundary condition:

$$u(x,0) = u_t(x,0) = 0$$

$$u(0,t) = u(a,t) = 0, 0 < x < a, t > 0$$

I am going to apply Laplace transform with respect to t then. So, when I do that I get the following expression:

$$\frac{d^2U(x,s)}{dx^2} - \frac{s^2}{c^2}U(x,s) = \frac{k}{s}\sin\left(\frac{\pi x}{a}\right)$$

So, if I were to solve this. So, this is an ODE this is a second order inhomogeneous ODE. So, this will have two solutions in particular it will have a homogeneous solution and it will have a particular solution. So, the homogeneous solution u h is found by a taking the right hand side of this equation to be 0 and we find that the homogeneous solution looks is of this form :

$$u = u_n + u_p$$

$$u_h = Ae^{\left(\frac{sx}{c}\right)} + Be^{-\left(\frac{sx}{c}\right)}$$

$$u_p = \frac{k\sin\left(\frac{\pi x}{a}\right)}{a^2 s\left[s^2 + (\frac{\pi c}{a})^2\right]}$$

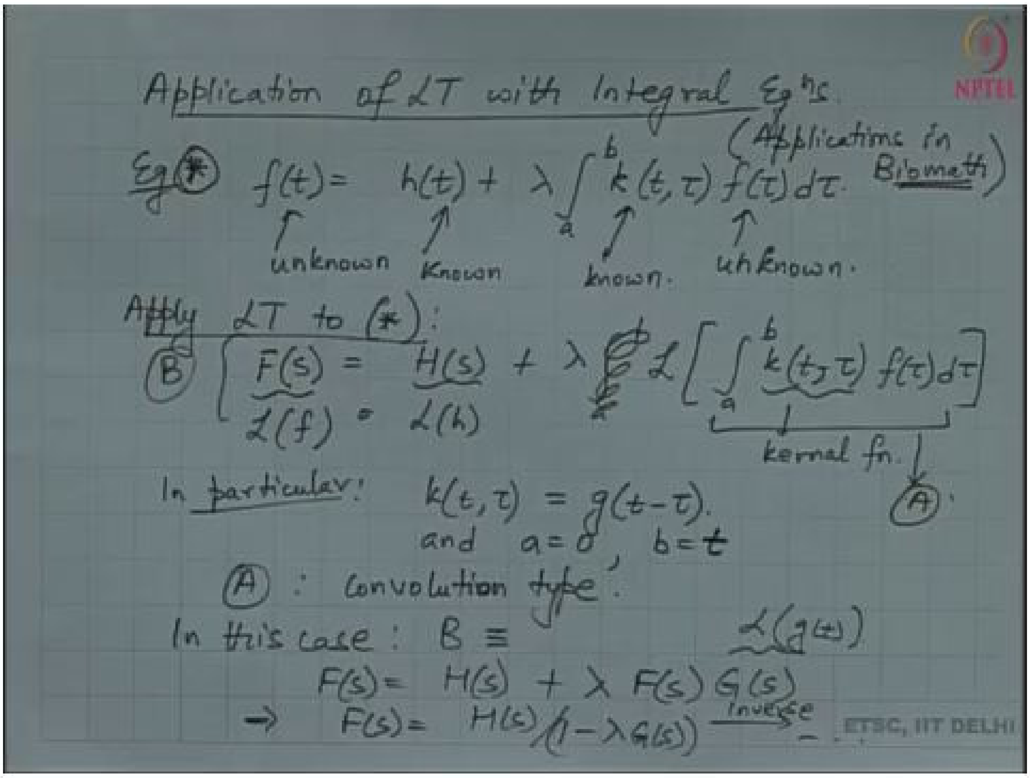
Applying boundary conditions,$u_n \equiv 0 \quad (A, B \equiv 0)$

$$u(x,s) = \frac{k}{(nc)^2} \sin\left(\frac{\pi x}{a}\right) [\frac{1}{s} - \frac{s}{s^2 + (\frac{\pi c}{4})^2}$$

So, let me just apply inverse transform to get the solution

$$u(x,t) = \frac{k}{(\pi c)^2} \sin\left(\frac{\pi x}{a}\right) [1 - \cos\frac{\pi c}{a} t]$$

So, that is the answer to this solution to this problem .



Let us continue our discussion on applications of on applications or example problems in Laplace transform with equations being of the integral type ok. So, we will see that the equations that we are dealing with are of this form :

$$f(t) = h(t) + \lambda \int_a^b k(t,\tau) f(\tau) d\tau$$

So, these types of integral equation applications frequently arises in problems related to mathematical biology. Specially we will see that you know in problems of you know particle aggregation or you know in blood flow simulations we frequently encounter problems involving these integral equations and we will see that the function apart from the solution to the problem inside the integral is the well known the Kernel function. So, moving back to this solution to this Laplace equation to this equation we see that. So, these have wide ranging applications in biomathematics ok. So, let us now look at one situation. So, if we were to apply let us say this is my example let us say example star. So, if we were to apply Laplace transform to star what I see is the following transformed equations:

$$F(s) = H(s) + \lambda L\left[\int_a^b k(t,\tau)f(t)d\tau\right]$$

$$k(t,\tau) = g(t-\tau)$$
$$\text{and } a = 0, \quad b = t$$

So, A is of convolution form. So, it is of the convolution type right. So, if we replace these limits of the integral and replace the kernel we see that this integral defined by this argument is nothing, but the convolution of two functions. So, in this case my equation let us say this is my equation B my B reduces to the following.

$$B \equiv F(s) = H(s) + \lambda F(s)G(s)F(s) = H(s)(1-\lambda G(s))$$

Where G of s is the Laplace transform of g of t or I get that F of s is H of s divided by 1 minus lambda times G of s by just taking all the coefficients of F of s on one side and then we can take an inverse of this function to find the solution to the problem ok. So, let us look at an example.