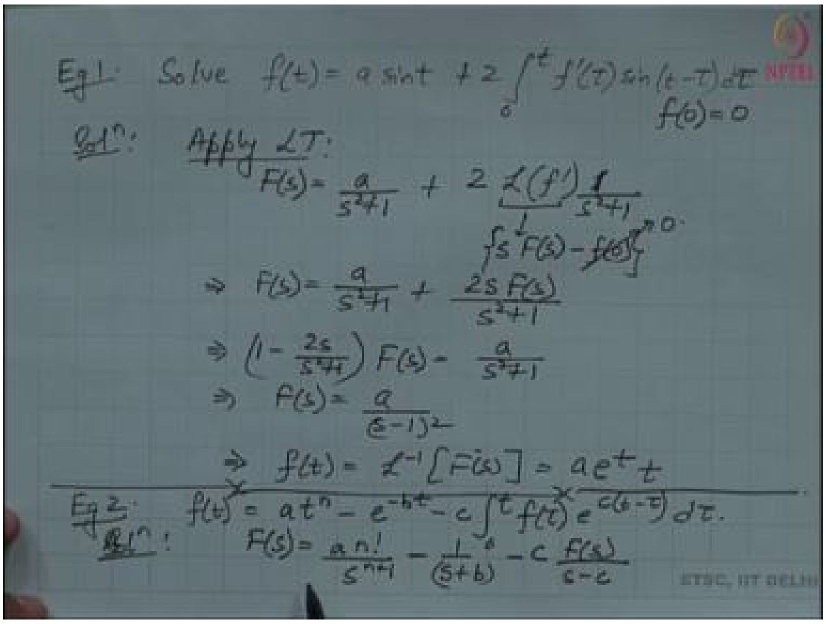
Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture –07
Applications of Laplace Transforms (Continued) Part 2



Example 1:

$$\text{Solve } f(t) = a \sin t + 2 \int_0^t f'(\tau) \sin(t - \tau) d\tau$$
$$\text{and} \quad f(0) = 0$$

Solution:

Apply Laplace Transform:

$$F(s) = \frac{a}{s^2+1} + 2L(f\prime)\frac{1}{s^2+1}$$

$$L(f)\prime = sF(s) - f(0)$$

$$F(s) = \frac{a}{s^2+1} + \frac{2sF(s)}{s^2+1}$$

$$\Rightarrow \left(1 - \frac{2s}{s^2+1}\right) F(s) = \frac{a}{s^2+1}$$

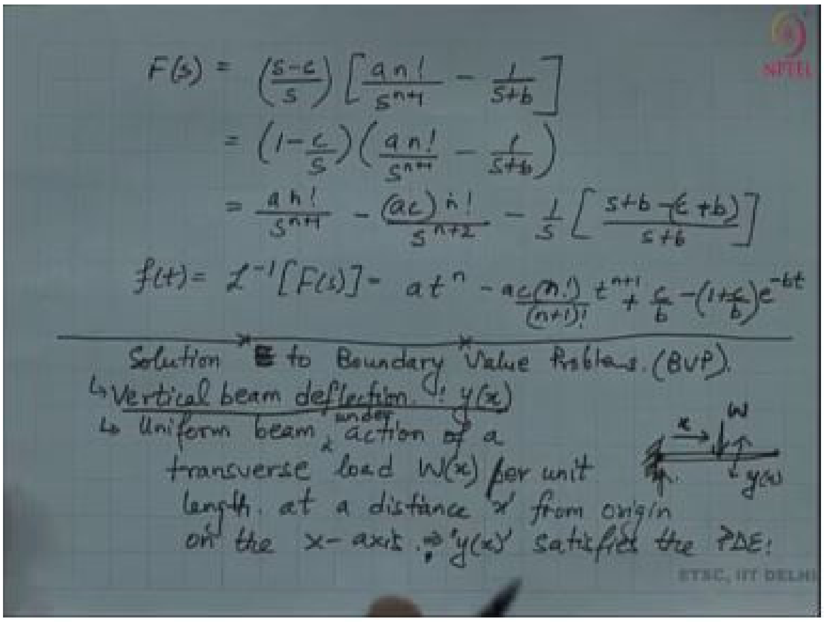
$$\Rightarrow f(s) = \frac{a}{(s-1)^2}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = ae^t t$$

Moving on let us look at one more example. So, I am given that:

Example 2:

$$f(t) = at^n - e^{-bt} - c\int_0^t f(\tau)e^{c(t-\tau)}d\tau$$



Solution:

$$F(s) = \frac{an!}{s^{n+1}} - \frac{1}{(s+b)} - c\frac{f(s)}{s-c}$$

$$F(s) = \left(\frac{s-c}{s}\right)\left[\frac{an!}{s^{n+1}} - \frac{1}{s+b}\right]$$

$$= \left(1-\tfrac{c}{s}\right)\left(\tfrac{an!}{s^{n+1}} - \tfrac{1}{s+b}\right)$$

$$= \frac{an!}{S^{n+1}} - \frac{(ac)n!}{s^{n+2}} - \frac{1}{s}\left[\frac{s+b-(c+b)}{s+b}\right]$$

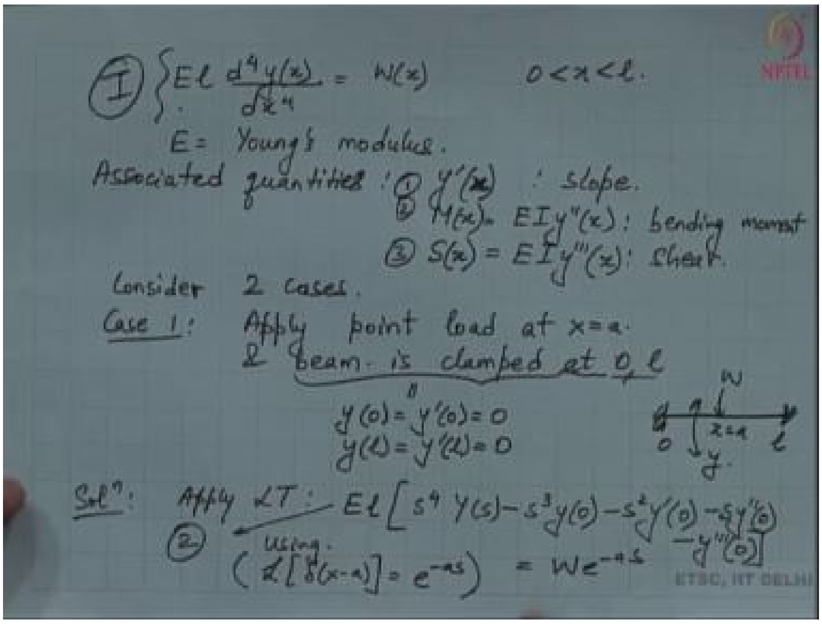
$$f(t) = \mathcal{L}^{-1}[F(s)] = at^n - \frac{a(n!)}{(n+1)!}t^{n+1} + \frac{c}{b} - (1+\frac{c}{b})e^{-bt}$$

So, then I have one more situation, I need to discuss about the solution to, some boundary value problems, using Laplace transform or BVP. So, one of the popular examples in this situation is the solution to the beam equation. So, let me just mentioned what is this beam equation. So, suppose we have a rigid rod and the rod is hinged to the wall and we see that, if the rod is hinged it could be either hinged at 1 end or suppose the rod is hinged at 2 ends, then and on top of that, on this beam we apply some homogeneous load or a point load.

Or a load in a particular region of this beam, hence and at let us say at time t equal to 0 or. So, and then we see, what happens to this beam, after this load is removed or perhaps due to the application of certain other conditions, which we will see in this example. So, let us move ahead. So, we see that, let us say I have I denote my vertical beam deflection, vertical deflection right. So, I have a beam here. So, let us say this is a wall and the beam is as follows and it is hinged here in the wall, it is hinged here and then I put a load, let us say the load is put homogeneously or at a point or in the certain region but, some load is put and then we see how does the position of the beam changes.

So, we could have different sorts of condition at this wall and we are going to describe the solution for some of these cases. So, let us say this vertical deflection is denoted by y(x), where x is this variable along the beam. So, the vertical beam deflection is denoted by y of x and the beam is it is a uniform; it is a uniform beam meaning that the density of the beam is uniform and it is being acted upon. So, under the action, it is being acted upon a transverse load. Let us say the load of the form w x, per unit length . So, let us say at a distance. So, this x denote the distance at which this load is applied so, at a distance x from origin. So, we fix our coordinate system and see where is this load applied so, it is being applied at a distance x from the origin of the fixed frame.

So, origin on the X axis and so, we see that the vertical deflection the vertical deflection y. So, y(x) it implies y(x) satisfies, the PDE of the following form:



$$= E\ell \frac{d^4 y(x)}{dx^4} = W(x) \quad 0 < x < \ell$$
$$E = \text{ young modulus.}$$

Associated quantities:

$$y'(x) = \text{slope}$$
$$M(x) = EIy''(x): \text{ bendiny moment}$$
$$S(x) = EIy'''(x): \text{ Sheat.}$$

So, we could consider, 2 scenarios, consider 2 scenarios, 2 cases. So, we can think of case 1, where we apply a point load, apply a point load at:

Consider 2 cases:

case 1:

Apply point load at $x = a$.
and beam is clamped at $0, l$
$y(0) = y'(0) = 0$
$y(l) = y'(l) = 0$

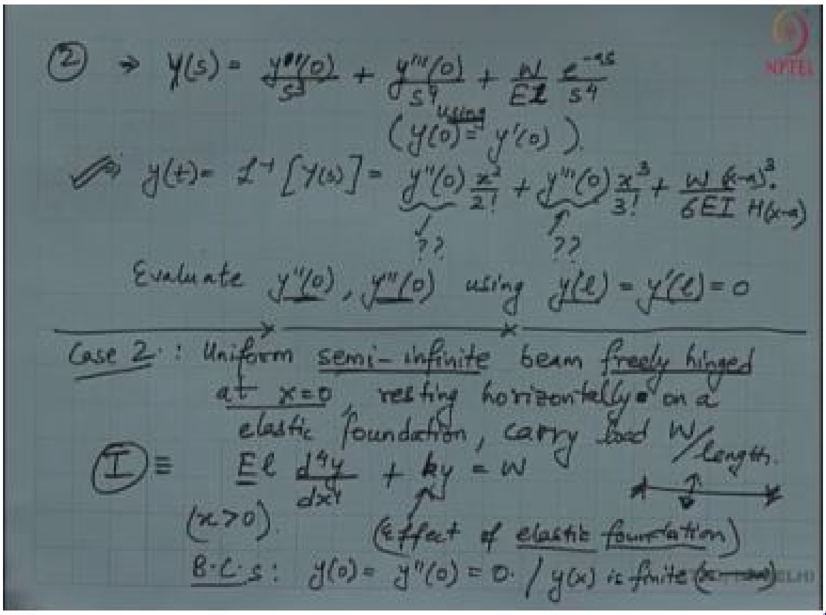
So, the it is a Laplace transform a point load will be given by,:

Solution:

Apply Laplace Transform:

$$El\,[s^4 y(s) - s^3 y(0) - s^2 y'(0) - s y''(0)\ .$$
$$y'''(0)]) = we^{-as}$$

$$\text{using: } L[\delta(x-a)] = e^{-as}$$



$$(2)\ \Rightarrow y(s) = \frac{y'(0)}{s^3} + \frac{y'''(0)}{s^4} + \frac{w}{El}\frac{e^{-as}}{s^4}$$

$$y(t) = L^{-1}[y(s)] = y''(0)\frac{x^2}{2!} + y'''(0)_{32}\frac{x^3}{3!} + \frac{w(x-a)^3}{6EI}H(x-a)$$

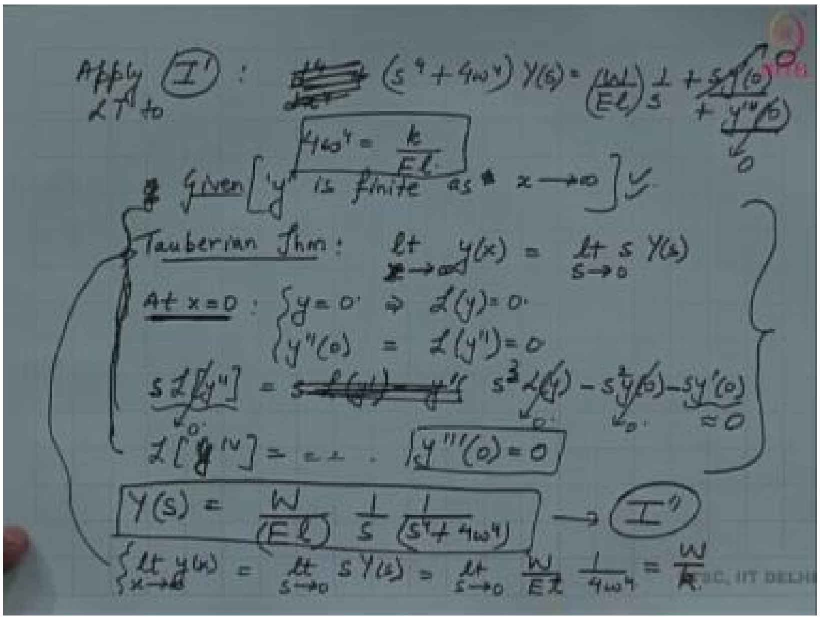
$$\text{Evaluate } y''(0)\,, y'''(0) \ \text{ using } y(l) = y'(l) = 0$$

So, what this extra terms does is that, when we have a beam, when we provide a load to the beam, the beam is going to deflect, due to elasticity of this beam here is elasticity affect, the beam is going to deflect up and down up and down right. So, there is a deflection of the beam from one position to the other. So, it is a vibration now, when we have this elastic affect, this elasticity tries to damp, these vibrations and we get this sort of a term, which is the damping term.

So, moving ahead so, of course, since this being a semi - infinite beam:

$$1 = \begin{array}{l} \text{El}\,\frac{d^4y}{dx^4} + ky = w \\ ,(x > 0) \end{array}$$

$$\text{B.C.S:} \quad y(0) = y''(0) = 0$$



Apply 1 to Laplace Transform:

$$\left(s^4 + 4w^4\right)y(s) = \left(\frac{w}{El}\right)\frac{1}{s} + \frac{sy'(0)}{y'''(0)}$$

Given:

$$4w^4 = \frac{k}{El}$$

Tauberian Theorem:

$$\lim_{x\to\infty} y(x) = \lim_{s\to 0} sy(s)$$

I see that some of these terms these vanishes otherwise, this result will not hold. So, to see the effect let me just quickly show you.

$$\text{At } x = 0: \left\{ \begin{array}{l} y = 0 \to L(y) = 0 \\ y''(0) = L\left(y''\right) = 0 \end{array} \right.$$

$$sL[y''] = s^3L(y) - s^2y(0) - sy'(0) = 0$$

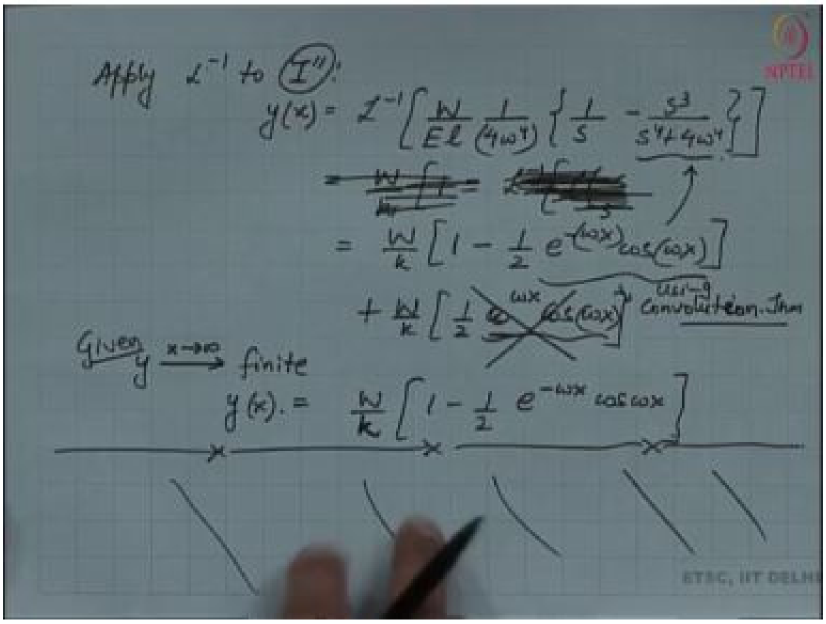
So, again we write the expression for this and deduce our argument using these results and also the given boundary condition. So, when so, using these arguments I see that these 2 are 0 so, moving ahead my solution,

$$y(s) = \frac{w}{(El)}\frac{1}{s}\left(\frac{1}{s^4 + 4w^4}\right)$$

$$\lim_{x\to\infty} y(x) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}\frac{w}{Et}\frac{1}{4w^4} = \frac{w}{k}$$

Apply L^{-1}

$$y(x) = L^{-1}\left[\frac{w}{El}\frac{1}{(4w^4)}\left\{\frac{1}{s} - \frac{s^3}{s^4+4w^4}\right\}\right]$$



$$= \frac{w}{k}\left[1 - \frac{1}{2}e^{-wx}coswx\right] + \frac{w}{k}$$

So,

$$y(x) = \frac{w}{k}\left[1 - \frac{1}{2}e^{-wx}\cos(wx)\right]$$

Now, we also so, this is the second expression of this solution, the only problem is that we know that y when x goes to infinity, y is some finite value and I know that this value is not .