Integral Transforms and Their Applications
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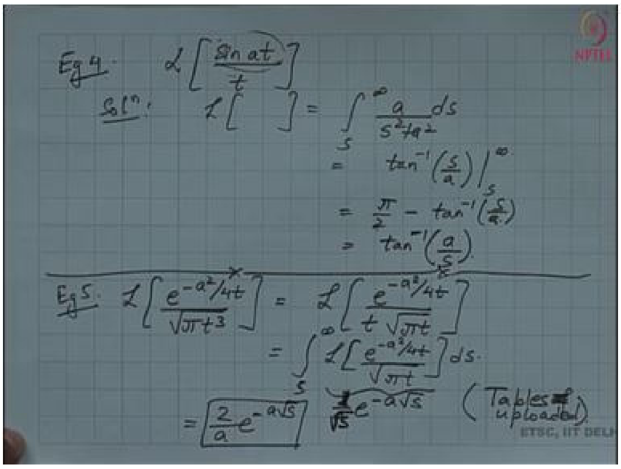
Lecture – 5
Inverse Laplace Transforms, Initial and Final Value Theorems - 02

Example (5): Find

$$\mathcal{L}\left[\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}\right] =?$$

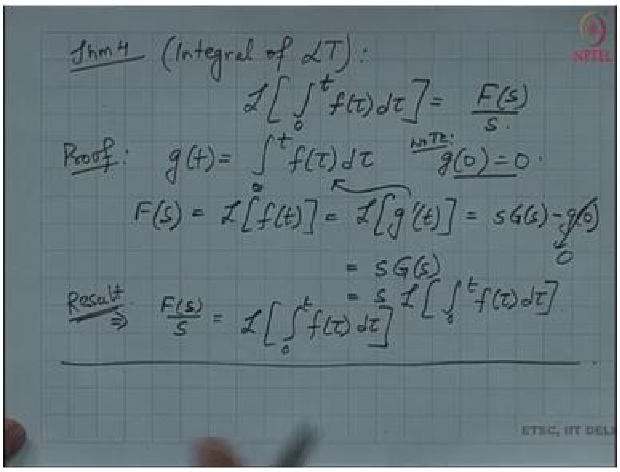


Solution:

$$\mathcal{L}\left[\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}\right] = \mathcal{L}\left[\frac{e^{-a^2/4t}}{t\sqrt{\pi t}}\right]$$

$$= \int_S^\infty \mathcal{L}\left[\frac{e^{-a^2/4t}}{\sqrt{\pi t}}\right] ds$$

$$= \frac{2}{a}e^{-a\sqrt{s}}$$



Theorem 4: So, moving on I have another result for the integral of the Laplace transforms. So, let it let me state that result in terms of a theorem. So,I have another result in terms of the integral of the Laplace transforms, so let me state it in the form of a theorem. So, the theorem says that

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

Proof :

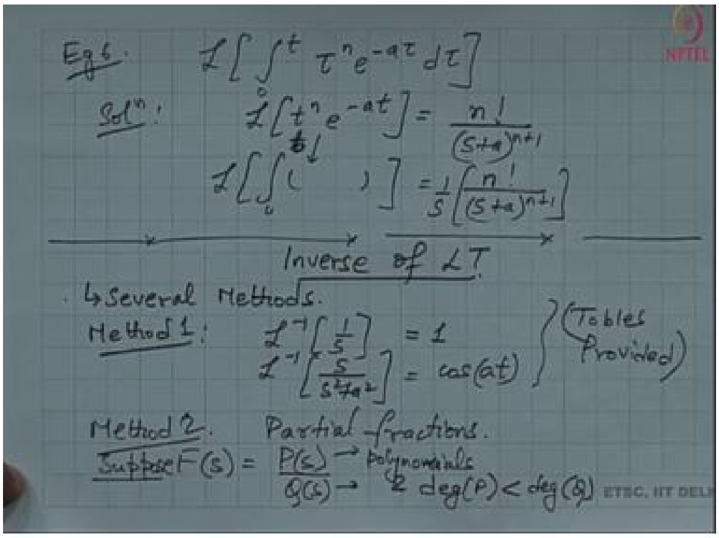
$$g(t) = \int_0^t f(\tau)d\tau$$

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[g'(t)] = sG(s) - g(0)$$

$$= SG(S)$$
$$= s\mathcal{L}\left[\int_0^t f(\tau)d\tau\right]$$

Result:

$$\frac{F(s)}{s} = L\left[\int_0^t f(\tau)d\tau\right]$$



Example(6): Find

$$\mathcal{L}\left[\int_0^t \tau^n e^{-a\tau} d\tau\right]$$

Solution: So, applying the theorem above what I see is that,

$$\mathcal{L}\left[\ t^n e^{-at}\ \right] = \frac{n!}{(s+a)^{n+1}}$$

$$L\left[\int_0^\infty t^n e^{-at}\right] = \frac{1}{s}\left(\frac{n!}{(s+a)^{n+1}}\right)$$

Inverse of Laplace Transform: So, then let me start another topic of discussion that is to find the inverse of my Laplace transform ok. So, let me outline several methods; so there are several methods I am going to outline each one of them. So, let me just outline the easiest one.

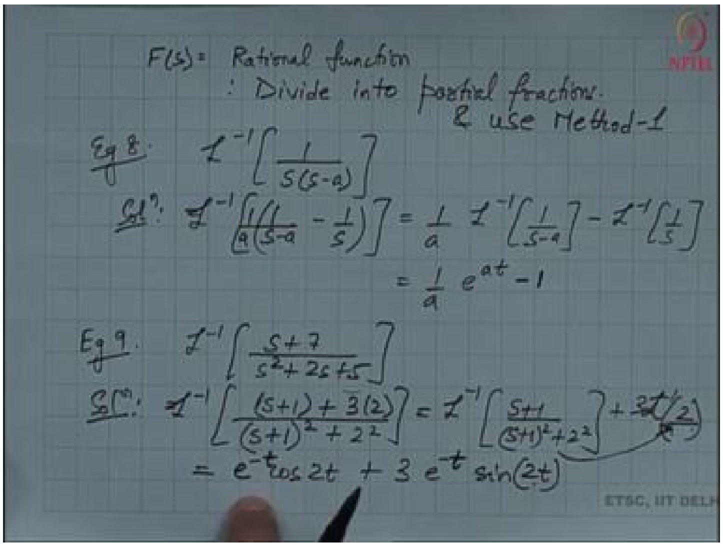
Method 1:

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$$

Method 2:The other way could be the method of partial fractions; so what is this method? So, suppose my Laplace transform of a function. So, suppose I have that the function

$$F(s) = \frac{P(s)}{Q(s)} \qquad \text{Also}, \deg(P) < \deg(Q)$$



So, then the idea is; so we are looking at a rational function; so F(s) is a rational function. So, what I do is; I divide my rational function into partial fractions and then use the method that I have used in method 1. So, use the table that I have provided.

Example (8):

$$L^{-1}\left[\frac{1}{s(s-a)}\right]$$

Solution:

$$\mathcal{L}^{-1}\left[\frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right)\right] = \frac{1}{a}\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] - L^{-1}\left[\frac{1}{s}\right]$$

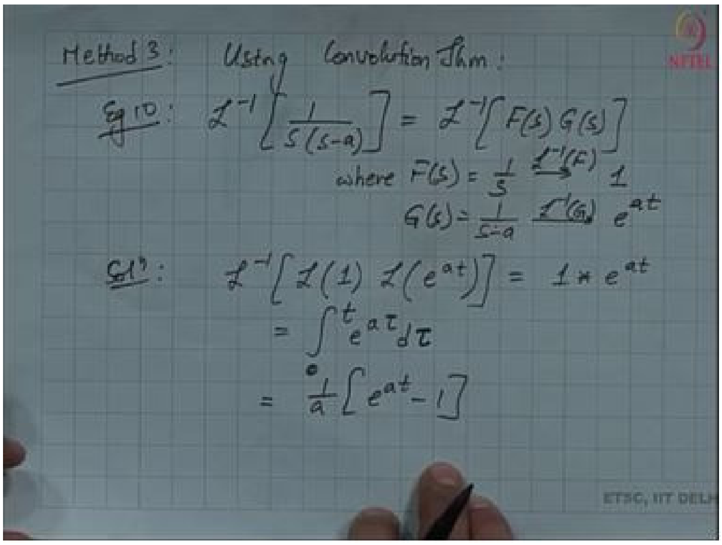
$$= \frac{1}{a}e^{at} - 1$$

Example (9):

$$\mathcal{L}^{-1}\left[\frac{s+7}{s^2+2s+5}\right]$$

Solution:

$$\left[\frac{(s+1)+3(2)}{(s+1)^2+2^2}\right] = \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + 3\mathcal{L}^{-1}(2)$$
$$= e^{-t}\cos 2t + 3e^{-t}\sin(2t)$$



Method 3:So, then another way to find the inverse is by the method of convolution. So, let me call that this, is my method 3. So, using convolution theorem ok; so what I see is that again let me try to solve the same problem that I did in example 9 using the method of convolution. So, I have to find the inverse of this function. So, notice let me write it in the form of the product of two Laplace transforms

Example (10):

$$\mathcal{L}^{-1}\left[\frac{1}{s(s-a)}\right] = \mathcal{L}^{-1}[F(s)G(s)]$$

where,

$$F(s) = \tfrac{1}{s} \xrightarrow{\mathcal{L}^{-1}(F)} 1$$
$$G(s) = \tfrac{1}{s-a} \xrightarrow{\mathcal{L}^{-1}(G)} e^{at}$$

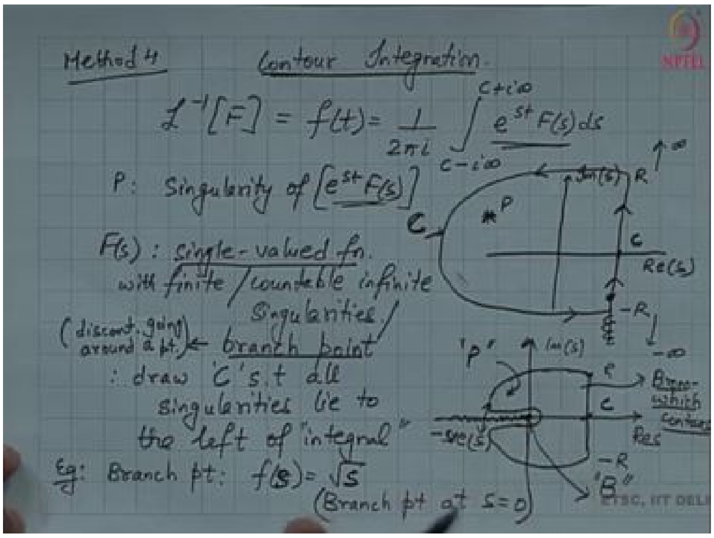
Solution:

$$\mathcal{L}^{-1}\left[\mathcal{L}(1)\mathcal{L}\left(e^{at}\right)\right] = 1 * e^{at}$$

$$= \int_0^t e^{a\tau}\,d\tau$$

$$= \frac{1}{a}\left[e^{at}-1\right]$$

So, that is how we use a method of convolution. So, then another well; so these three methods have been typically for most of the simple cases where we are studying the Laplace transform for and inverse transform for simple functions. So, in general when we have a very complicated function inside this integral of the Laplace transform and we have to evaluate the inverse; I need to use the method of contour integrals. So, let me just highlight what is this method.

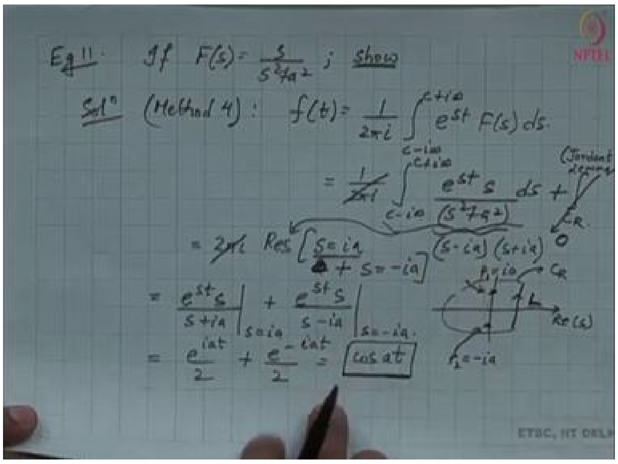


Method 4: So, the inverse this was earlier described that the inverse of a Laplace transform; it was earlier described in one of my lectures the definition of the inverse follows from this integral. So, the integral is defined by,

$$\mathcal{L}^{-1}[F] = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds$$

So, now I am going to describe what is this integral how does this integral looks like?(Refer the slide for this part): So, notice that C is a finite number. So, C is the real number and this integral tells me that this sort of an integration is over a complex plane. So, if I were to look at the s plane here. So, let me say that this is my real s and this is my imaginary s. So, over the s plane I see that my real s varies from C; so, the real s takes of finite value let us say at C and the imaginary s goes from $-\infty to \infty$. So, we are evaluating this line integral for this transform, for this inverse transform.

Now, so one way is to evaluate this line integral which is most of the time not very simple to evaluate. The other way is to complete to change this line integral into a contour integral. So, that let us say I have some singularities of this function. So, let us say I have a singularity at a point P, where P is a let us say a singularity of the function $e^{st}F(s)$; the function inside this integral. And what I do is I try to complete this; this change this line integral into a contour integral which encompasses this singularity right. So, instead of this line integral now I am going to evaluate this contour integral and let us say my values over the imaginary axis is from minus R to R and R goes to $-\infty$ So, as we can see that as we increase the size of this the domain encompassed by this contour, we are basically evaluating this one over; this integral over the value that we want that is the inverse. So, the integral the line integral over this straight line is the value that we want the inverse transform. And if we if we can somehow show that the value of the integral over this contour over this curved contour, over this curved line is 0 then I can straight away use my residue theorem. To say that the value of this integral over this contour is equal to the residue of this this function at this point of singularity; so that is the whole idea. So, let us say that let us say that F(s) is a single valued function just to make it a bit more simplified situation; so single valued function. And f has either finite or countable infinite singularities or it may also have the so called what I defined by branch points. So, what are these branch points? Now it turns out that at certain value of this function; so let me just redraw this contour to show you an example of a branch point. So, let us say that I have the same contour and let me just draw this. So, I have real of s and this one y axis will be the imaginary of s. So, I have this line integral that I want; so, from R to minus R to R and this real value C. And I were to were to complete the contour notice that let us say my singularity of this function lies to the left of this contour this; this integral. So, let me just say I am going to draw the contour in such a way this curve; close contour in such a way that draw let me say that this contour is C. So, draw C such that all singularities lie to the left of the integral that we are trying to evaluate. So, all my singularities are to the left of the integral that we are trying to evaluate. Now besides singularity suppose we have a branch point; so what is branch point? Now notice let us take let us take a function f of z. So, I am going to just give you a highlight of what is branch point. So, consider an example of function f of z to be well in this case F(s) to be $\sqrt{s}$. So, I know that $\sqrt{s}$; well in general when s becomes negative then the square root function becomes purely imaginary and if s is positive then the square root function is completely real or you know. So, in the complex plane we see that at s equal to 0 there is a sudden jump from this positive half of the complex plane to the negative half of the complex plane. So, we see that around this axis around the negative S axis negative S real S axis there is a sudden jump of the value of the function from being real from being real, but positive imaginary part to real with negative imaginary part. So, the argument of this complex number jumps from say θ to $\theta + \pi$. So, this sort of a with the jump we call this as the branch point. So, the value of S there is shows a sudden jump from one value to the other value; so there is a discontinuity. So, I can say that branch points are nothing, but discontinuity; so these are discontinuities around; well going around a particular point going around a point ok. So, in this case we see that the branch point here is at s equal to 0. So, which means we have to if we were to define a single valued function a single valued function then we have to avoid this this branch cut where the function may attain multiple values or there is a change in the value of the function. So, I am to complete the contour; I am going to evade this branch this, these values where there is a sudden jump and I am going to complete my contour in this fashion ok. So, this is my branch point let us say that this is B and this is at P ok. So, this sort of a contour this particular contour that I have drawn which evaluates this integral here is also known as the Brom-which contour let me just write down that this is the Brom-which contour. So, students who are more interested to look at Brom-which contours should read a standard text in complex variables which will highlight how to evaluate integrals over Brom-which contour and we are going to see some examples in this lecture as well ok.



Example (11):

$$\text{If } F(s) = \frac{s}{s^2 + a^2}; \quad s \text{ shem}$$

Solution:(Method 4)

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds$$

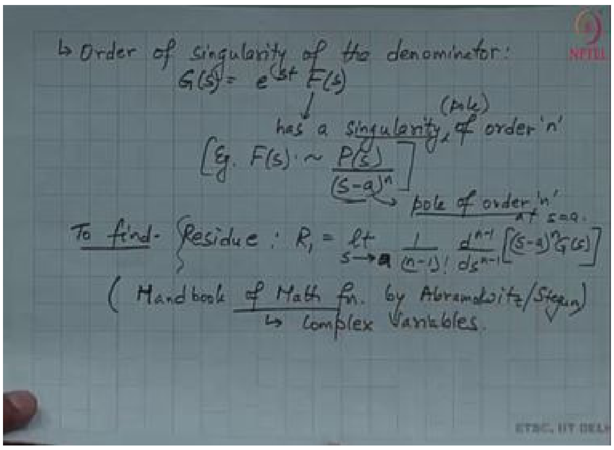
$$= \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} \frac{e^{st} s}{(s^2 + a^2)} ds + \int_{C_R} (.)$$

$$= 2\pi i \,\mathrm{Res}\,[s = ia, +s = -ia,]$$

$$= \left.\frac{e^{st} s}{s + ia}\right|_{s=ia} + \left.\frac{e^{st} s}{s - ia}\right|_{s-ia}$$

$$= \frac{e^{iat}}{2} + \frac{e^{-iat}}{2} = \cos at \tag{1}$$

Now let me just say that we see that I need to find that to find the residue I need to see what is the order of the singularity in the denominator?



So order to find the residue I need to see the order of the singularity of the denominator ok. So, let us say if I have

$$G(s) = e^{st}F(s) \tag{2}$$

my F(s) is function has a singularity of order n. Let us say; so let us say my F(s) is of this form.

$$F(s) \sim \frac{P(s)}{(s-a)^n} \tag{3}$$

So, we see that there is a factor of s-a which is appearing n times.So, the singularity is of the order n; say we call this as a pole singularity or I also call this as a pole of order n. So, this is a pole of order n at S equal to a. So, to find the residue I need to just find it using the following formula. So, again these are all standard complex variables techniques, students are again asked to find these results in a standard complex variables textbook which is given in my course webpage. So, the residue R 1 is :

$$\text{Residue}: \ R_1 = \lim_{s\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left[(s-a)^n G(s)\right] \tag{4}$$

So, please look at my book of handbook of Math Functions by Abramowitz and Stegan for more complex variables results ok. So, this result is directly taken from a standard complex variables textbook. So now which means; so I am going to use this result to find out my contour integral for the inverse Laplace transform, when I have poles of order n; so let me give you an example.