Integral Transforms and Their Applications
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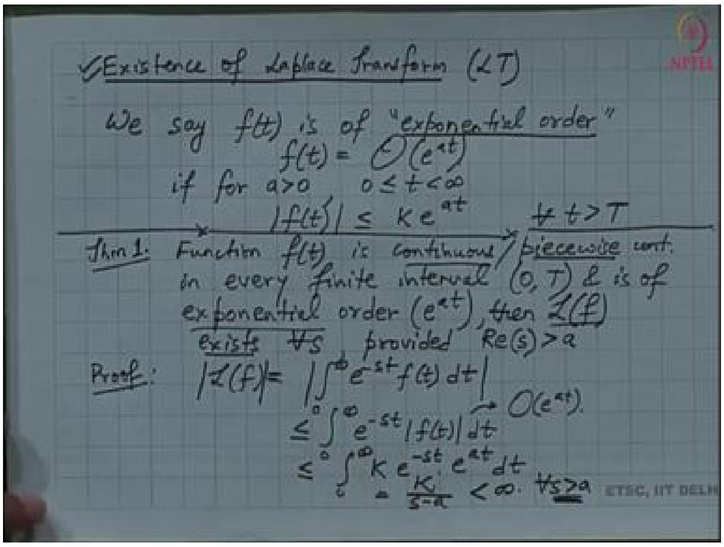
Lecture – 5
Inverse Laplace Transforms, Initial and Final Value Theorems - 01

So, good morning everyone, so in the last lecture we saw the basic properties of Laplace transform. We had introduced Laplace transform and we also saw; what is the Laplace transforms of a derivative of a function to any order. So, in today's lecture I am going to continue my discussion on Laplace transform especially I will introduce some more properties, as well as few important results including the convolution of two Laplace transforms, as well as how to evaluate the inverse transform of the Laplace transform.



So, today I will start the lecture today by introducing the convolution of the convolution of the Laplace transform and let me just right away introduce the property by introducing a result in the form of theorem. So, I call this as my convolution theorem which tells that if I have that my Laplace transform of a function is F(s) and the Laplace transform of another function G(s). Then the theorem says that the Laplace transform of $f * g$ is given to be the Laplace transform of F times the Laplace transform of G. So, which means;

$$\text{If, } \mathcal{L}(f) = F(s), \quad \mathcal{L}(g) = G(s)$$

then,

$$\mathcal{L}[f * g] = F(s)G(s)$$

So, if am given that I am given the product of say two Laplace transforms and I want to evaluate the inverse of this product. Then it is equal to the convolution of the two function in the real domain and by convolution I define this finite integral as follows;

$$\mathcal{L}^{-1}[F(s)G(s)] = f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

So, let us look at some more properties of convolution in with relation to the Laplace transform.

Properties :

So, the first property says that the order of the convolution does not matter.

(1) $f * (g + b) = (f * g) * h$

(2) $f * g = g * f$

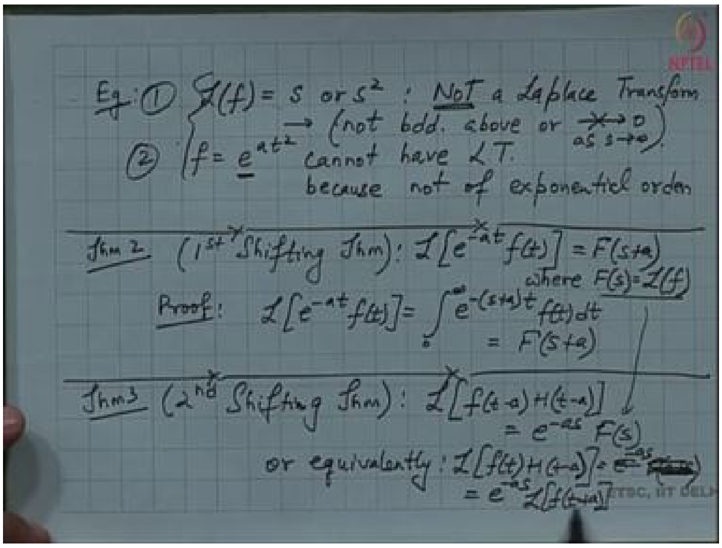
(3) $f * (ag + bh) = a(f * g) + b(f * h)$

(4) $f * (ag) = (af) * g = a(f * g)$

(5)
$$\mathcal{L}\left[f_1 * f_2 * ..... * f_n\right] = F_1(s)F_2(s)\ldots F_n(s)$$

(6)
$$\mathcal{L}\left[f^n\right] = -(\mathcal{L}[f])^n$$

So, let us look at some examples.



Example 1: Find
$$1 * \frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}$$

Solution:
$$1 * \frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}} = \int_0^t f(t-\tau)g(\tau)d\tau$$

$$= \int_0^t \frac{a}{2\sqrt{\pi}}\frac{e^{-a^2/4\tau}}{\sqrt{\tau^3}}d\tau$$

$$= \frac{aa}{2\sqrt{\pi}}\int_{\frac{a}{a\sqrt{t}}}^{\infty} e^{-x^2}dx(\frac{4}{a}) \quad \text{choose,} x = \frac{a}{2\sqrt{\tau}}$$

$$= \frac{a}{2\sqrt{\pi}}\left(\frac{4}{a}\right)\int_{\frac{a}{2\sqrt{t}}}^{\infty} e^{-x^2}dx$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_{\frac{a}{2\sqrt{t}}}^{\infty} e^{-x^2} \right]$$

$$= \frac{2}{\sqrt{\pi}} \left[1 - \int_{0}^{a/2\sqrt{t}} e^{-x^2} dx \right]$$

we know that,

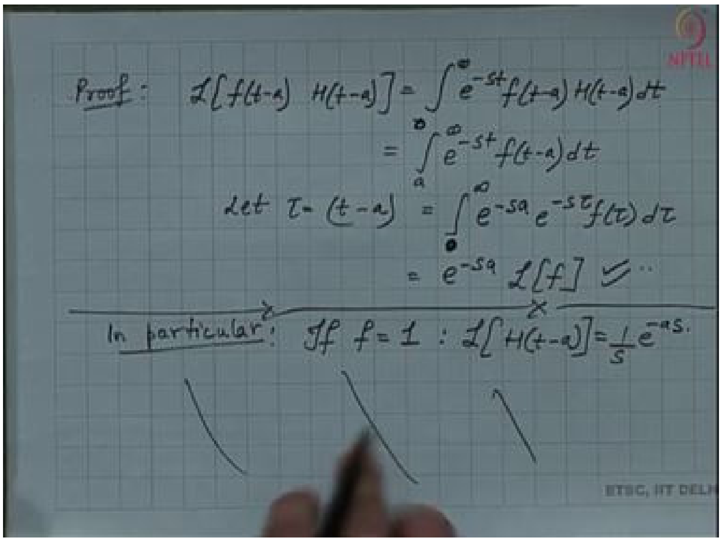
$$\mathrm{Erf}(x) = \int_{\infty}^{x} e^{-t^2} dt$$

So,

$$= \frac{2}{\sqrt{\pi}} \left(1 - \mathrm{Erf}\left(\frac{a}{2\sqrt{t}} \right) \right]$$

$$= \frac{2}{\sqrt{\pi}} \left[1 - \mathrm{Erf}\left(\frac{a}{2\sqrt{\tau}} \right) \right]$$

We know that, Erfc=1-Erf(x)



So,

$$\frac{2}{\sqrt{\pi}} \left[\mathrm{Erfc}\left(\frac{a}{2\sqrt{t}} \right) \right]$$

Example (2):Use convolution Theorem,Proove:

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Where,

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$\Gamma(m) = \int_0^\infty t^{m-1} e^{-t} dt \quad m > 0$$

Solution: Choose,

$$f(t) = t^{m-1} \quad (m > 0)$$

$$g(t) = t^{n-1}$$

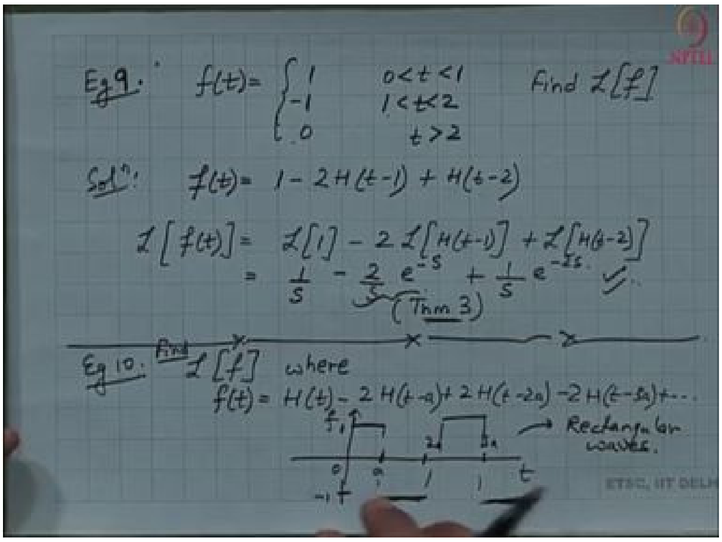
$$\mathcal{L}[f] = F(s) = \frac{\Gamma(m)}{S^m}$$

$$G(s) = \mathcal{L}(g) = \frac{\Gamma(n)}{s^n}$$

$$f * g = \int_0^t t^{m-1}(t-\tau)^{n-1} d\tau$$

$$= \mathcal{L}^{-1}[FG]$$

$$= \mathcal{L}^{-1}\left[\frac{\Gamma(m)\Gamma(n)}{S^{m+n}}\right]$$



$$f * g = \int_0^t \tau^{n-1}(t-\tau)^{n-1} d\tau = \Gamma(m)\Gamma(n)\mathcal{L}^{-1}\left[\frac{1}{s^{m+n}}\right]$$

$$\int_0^t \tau^{m-1}(t-\tau)^{n-1} d\tau = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} t^{m+n-1}$$

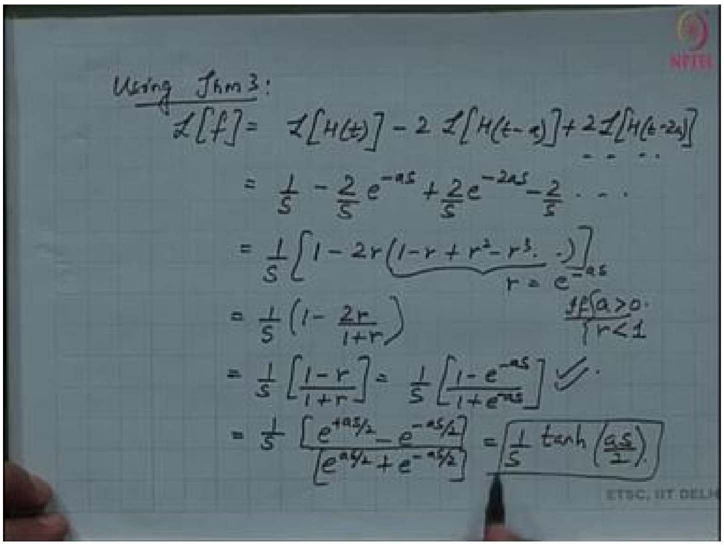
$$\text{Put } t = 1, \int_0^1 \tau^{m-1}(1-\tau)^{n-1} d\tau = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

So, next we look at some; so before moving on I need to just introduce one more result again without proof this time again. So, the result says that if I have that the function; I am given a function which is of exponential order, this following definition has already been introduced in one of my earlier lectures. Theorem (2):If f(t) is of exponential order as $t \to \infty$ then the Laplace transform,

$$\mathcal{L}[f] = \int_0^\infty e^{-st} f(t) dt \quad \text{is uniformly convergent}$$

Proof : Follows weistrass Test.

Please try to attempt the proof of this and you know. So, this result is going to be used in one of my later examples.



Properties: We know that,

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)dt$$

(a).Differentiation Property:

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{d}{dt}\left(e^{-st}\right)dt$$

$$= \int_0^\infty -te^{-st} f(t)dt$$

$$= \mathcal{L}[-tf(t)] = -\mathcal{L}[tf(t)]$$

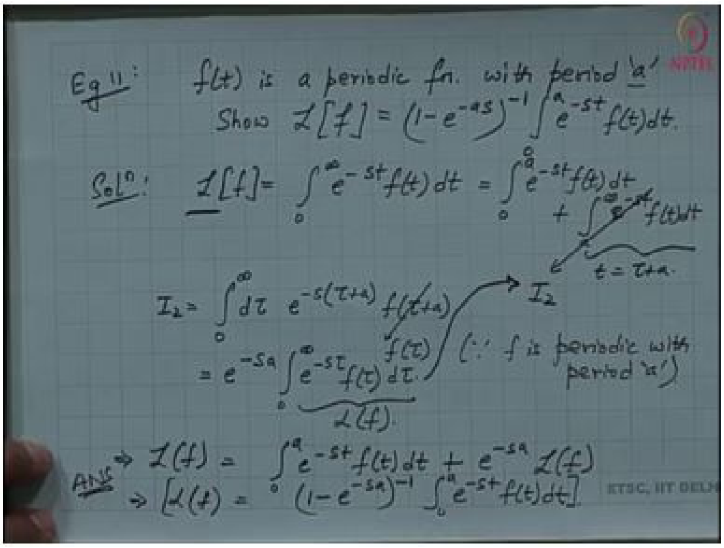
(b)

$$\frac{d^2F(s)}{ds^2} = (-1)^2\mathcal{L}\left[t^2 f(t)\right] \qquad \text{and goes to,}$$

(c)

$$\frac{d^nF(s)}{ds^n} = (-1)^n\mathcal{L}\left[t^n f(t)\right]$$

So, just let us see some quick examples of this property what I see is that;



Example (3):

(a)

$$\mathcal{L}\left[t^n e^{-at}\right] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\left[e^{at}\right]$$

$$= (-1)^n \left[\frac{1}{s+a}\right]^{(n)}$$

$$= (-1)^n \quad (-1)^n \frac{n!}{(s+a)^{n+1}}$$

$$= \frac{n!}{(s+a)^{n+1}}$$

(b)

$$\mathcal{L}[t \cos at] = (-1) \frac{d}{ds}\left[\frac{s}{s^2+a^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

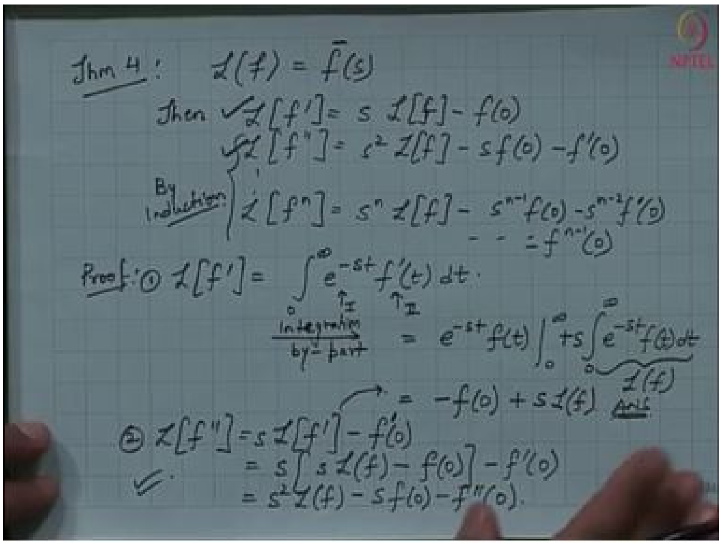
So, this is my Laplace transform of cos(at) and then of course, with a -1 outside. So, when I take the derivative; I am I get the following result. So, notice that one way to find the Laplace transform of this function is to use a standard definition and that definition will then involve integration by parts, but then the other way to utilize this Laplace transform is via this derivative formula.

So, then I have one more result;

Theorem(3):

$$\text{If } \mathcal{L}[f(t)] = F(s)$$

$$\text{then } \mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$$



Proof :

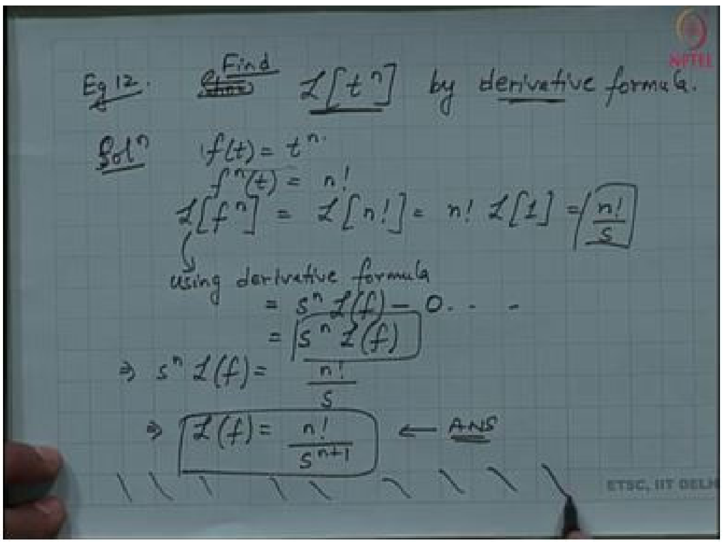
$$\int_S^\infty F(s)ds = \int_s^\infty ds \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\infty f(t)dt \int_s^\infty e^{-st}ds$$

$$= \int_0^\infty f(t)dt \left(\frac{1}{t}\right) e^{-st}$$

$$= \int_0^\infty \frac{f(t)}{t} e^{-st}dt$$

$$= \mathcal{L}\left[\frac{f(t)}{t}\right]$$



Example (4):Find

$$\mathcal{L}\left[\frac{\sin at}{t}\right]$$

Solution:

$$\mathcal{L}\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}ds$$

$$= \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \tan^{-1}\left(\frac{a}{s}\right)$$