Integral Transforms and Their Applications

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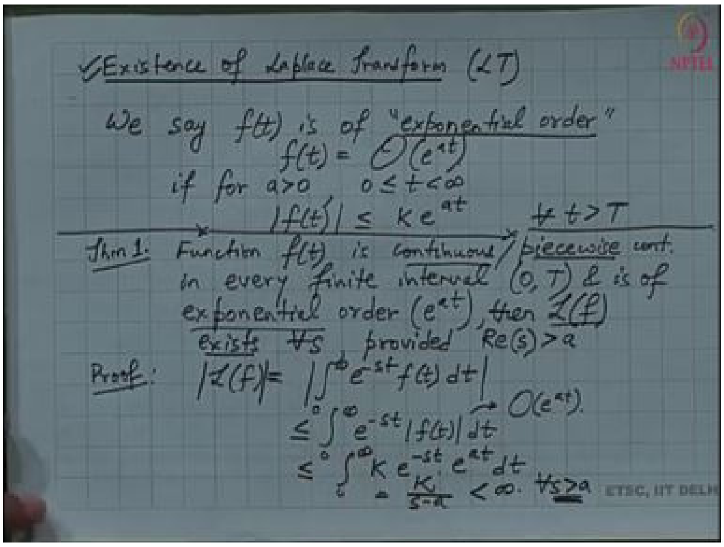
Lecture – 4
Applications of Laplace Transforms Part - 03

Before I move further, let me talk about the existence criteria.



Existence of the Laplace transform so, I am going to denote it by LT. So, we say so to define the existence criteria for Laplace transform, let me introduce what I call as the exponential order of a function the function being exponential order. We say that f of t is of is of exponential order exponential order also I denote it by following expression:

$$f(t) = \mathcal{O}\left(e^{at}\right)$$

$$\text{if, for} \quad a > 0, \quad 0 \leqslant t < \infty$$
$$|f(t)| \leq ke^{at} \quad \forall t > T$$

Essentially it is this definition is saying that a function is an x is of exponential order if it is bounded above by an exponential function.

So, in particular; so let us move on I have a nice result related to the existence of the Laplace transform. So, I denote it by theorem (1) here.

Theorem 1: So, it tells me that a function f(t) is continuous or suppose if it is continuous or piecewise continuous, in let us say in every finite interval 0 to T and is of exponential order, exponential order given by this function e^{at}, then my Laplace transform of this function f exists provided so this is for all s provided I have that the $Re(s) > a$ ok. So, to see the proof of this result let us quickly see what is the Laplace transform of f Proof:

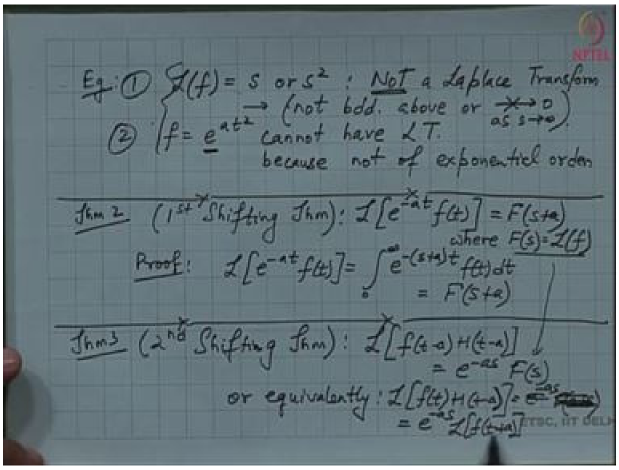
$$|\mathcal{L}(f)| = \left|\int_0^\infty e^{-st} f(t) dt\right|$$

$$\leqslant \int_0^\infty e^{-st} |f(t)| dt \quad \text{where,} \quad |f(t)| = \mathcal{O}(e^{at})$$

$$\leqslant \int_0^\infty ke^{-st}e^{at}dt$$

$$= \frac{K}{s-a} < \infty, \forall s > a$$

So, we see that if the function is of exponential order it is continuous or piecewise continuous then the Laplace transform exists.



So, to give you an example; so, to give you an

Example (1):

$$\mathcal{L}(f) = s \text{ or } s^2$$

So, this is not a Laplace transform; why is that? Because we see that this is not going to 0 as s goes to infinity. So, this is not bounded above ok. So, this is because this is not bounded above or does not go to 0 as s goes to ∞ ok. So, this is not a Laplace transform

Example (2):

$$f = e^{at^2}$$

So, this cannot have Laplace transform, why? Because this is not of the exponential order cannot have the Laplace transform here because not of exponential order in this scenario. So, Laplace transform cannot be found for each and every function here are two examples where I have shown that Laplace transform does not exist ok. So, in our previous result the theorem one was mainly a sufficient condition not a necessary condition. Now moving ahead let me just highlight few more theorems.

Theorem 2: 1st Shifting Theorem

$$\mathcal{L}\left[e^{-at}f(t)\right] = F(s+a)$$

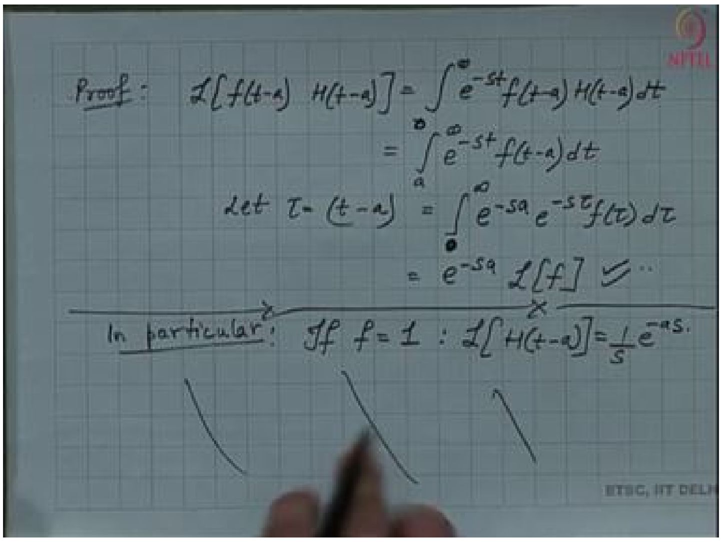
Proof :

$$\mathcal{L}\left[e^{-at}f(t)\right] = \int_0^\infty e^{-(s+a)t}f(t)dt = F(s+a)$$

Theorem 3: 2nd Shifting Theorem

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s) \quad \text{where, } F(s) = \mathcal{L}(f)$$

$$\text{or equivalently: } \mathcal{L}[f(t) + H(t-a)] = e^{-as}\mathcal{L}[f(t+a)]$$



Proof :

$$\mathcal{L}[f(t-a)H(t-a)] = \int_0^\infty e^{-st} f(t-a)H(t-a)dt$$

$$= \int_0^\infty e^{-st} f(t-a)dt$$

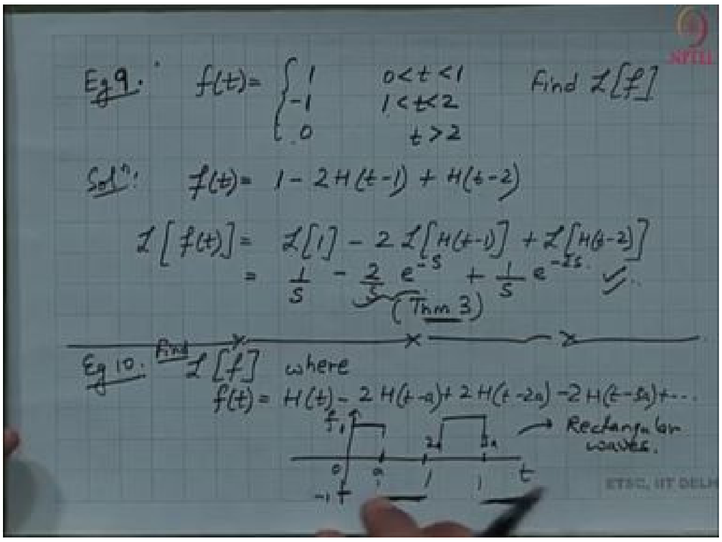
$$\text{Let}, \tau = (t-\alpha) = \int_0^\infty e^{-sa} e^{-s\tau} f(\tau)d\tau$$

$$= e^{-sa}\mathcal{L}[f]$$

So, in particular, I can use the theorem above the result above to find the Laplace transform of 1

$$\text{If,} \quad f = 1 : \mathcal{L}[H(t-a)] = \frac{1}{s}e^{-as}$$

So, let us look at some examples, let us look at some examples quickly.



Example (9):

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ 0 & t > 2 \end{cases} \qquad \text{Find } \mathcal{L}[f] = ?$$

Solution:

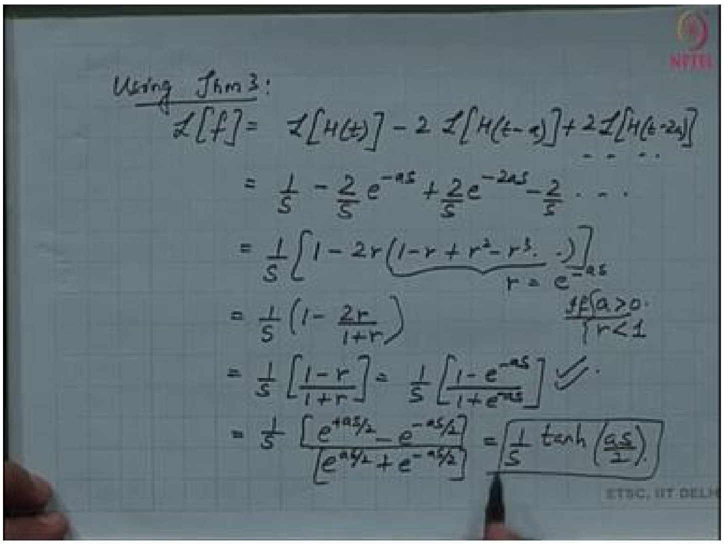
$$f(t) = 1 - 2H(t-1) + H(t-2)$$
$$\mathcal{L}[f(t)] = \mathcal{L}[1] - 2\mathcal{L}[H(t-1)] + \mathcal{L}[H(t-2)]]$$
$$= \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s}$$

So, then let us look at another example. So, now, you have to find the Laplace transform.
Example (10):

$$\text{Find } \mathcal{L}[f] \text{ where,} \quad f(t) = H(t) - 2H(t-a) + 2H(t-2a) - 2H(t-3a) + \cdots$$

So, if we see that if we try to plot this function let us say f versus t,(Refer the slide).

we see that if t is from let us say 0 to a. So, if t is from 0 to a, then all these Heaviside function vanishes and I get a 1.So, these are what I call as rectangular wave function; the rectangular waves we see that these are rectangles with time. So, I have to calculate the Laplace transform of this function. So, again this is quite straightforward using our theorem 3.



Using Theorem 3: $\mathcal{L}[f] = \mathcal{L}[H(t)] - 2\mathcal{L}[H(t-a)] + 2\mathcal{L}[H(t-2a)]....$

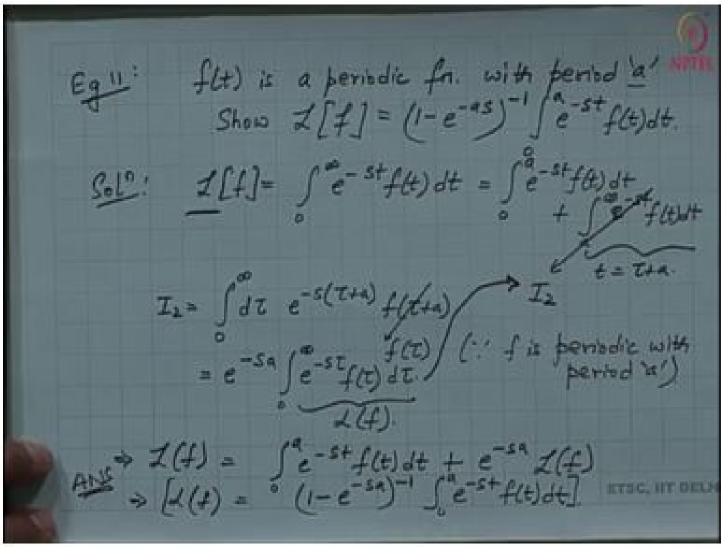
$$= \frac{1}{s} - \frac{2}{s}e^{-as} + \frac{2}{s}e^{-2as} - \frac{2}{s}\cdots$$
$$= \frac{1}{s}\left[1 - 2r\left(1 - r + r^2 - r^3..\right)\right]$$

$$= \frac{1}{s}\left(1 - \frac{2r}{1+r}\right) = \frac{1}{s}\left[\frac{1-r}{1+r}\right]$$

$$= \frac{1}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right] \quad \text{where,} r = e^{-as}$$

$$= \frac{1}{s}\left[\frac{e^{+as/2}-e^{-as/2}}{e^{+as/2}+e^{-as/2}}\right] = \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

Well the answer we have already got it here this is just a bit of simplification a bit more simplification to write it in compact form ok.



So, then I have another nice example I have another nice example.

Example(11): Given that f(t) is a periodic function it is a periodic function with period with period 'a' right and then. So, I need to show that.

$$\mathcal{L}[f] = \left(1-e^{-as}\right)^{-1}\int_0^a e^{-st}f(t)dt$$

Solution:

$$\mathcal{L}[f] = \int_0^\infty e^{-st}f(t)dt = \int_0^a e^{-st}f(t)dt + \int_a^\infty e^{-st}f(t)dt$$

$$I_2 = \int_a^\infty e^{-st}f(t)dt \quad \text{Let,} t = \tau + a$$

$$I_2 = \int_0^\infty d\tau e^{-s(\tau+a)}f(\tau+a)$$

Now notice that the last term $f(\tau+a)$ this is also equal to $f(\tau)$ because f is periodic f with period 'a' here ok.

$$= e^{-sa}\int_0^\infty e^{-s\tau}f(\tau)d\tau$$

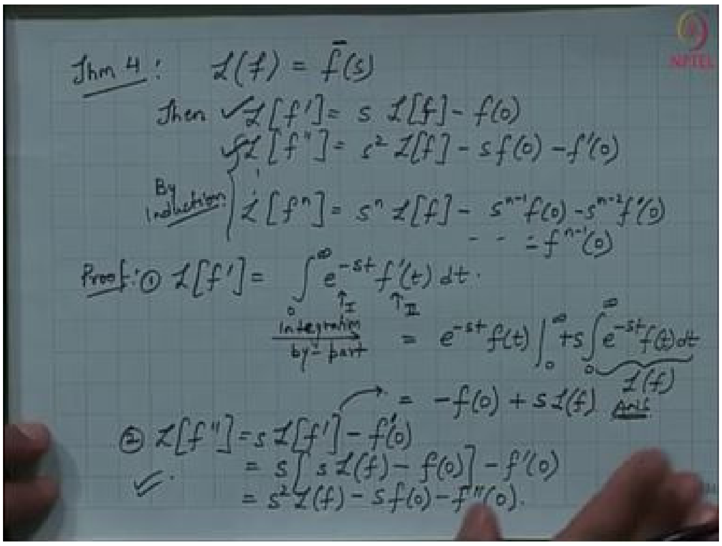
$$\text{Here,} \int_0^\infty e^{-s\tau}f(\tau)d\tau = L(f)$$

$$\mathcal{L}(f) = \int_0^a e^{-st}f(t)dt + e^{-sa}\mathcal{L}(f)$$

$$\left[\mathcal{L}(f) = \left(1 - e^{-sa}\right)^{-1} \int_0^{\alpha} e^{-st} f(t)dt\right]$$

This is my answer.

Before I end today's lecture, I want to also introduce another result and that is given in the form of a short theorem.



Theorem(4):

$$\mathcal{L}(f) = \overline{f}(s)$$

$$\text{Then, } \mathcal{L}\left[f\right] = s\mathcal{L}[f] - f(0)$$

By induction,

$$\begin{aligned}
&\mathcal{L}\left[f''\right] = s^2\mathcal{L}[f] - sf(0) - f'(0) \\
&\mathcal{L}\left[f^n\right] = s^n\mathcal{L}[f] - s^{n-1}f(0) - s^{n-2}f'(0) - ...... - f^{n-1}(0)
\end{aligned}$$

Proof :

$$(1) \quad \mathcal{L}\left[f'\right] = \int_0^{\infty} e^{-st} f'(t)dt$$

So, then I use integration by parts right

$$= e^{-st} f(t)\Big|_0^{\infty} + s\int_0^{\infty} e^{-st} f(t)dt$$

$$= -f(0) + s\mathcal{L}(f)$$

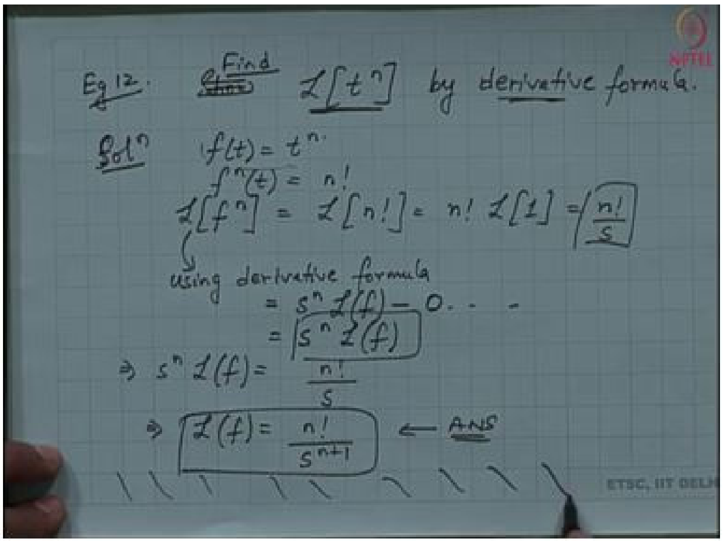
Notice that at infinity then the function is of exponential order, otherwise the Laplace transform would not have been defined. So, at infinity this expression will vanish.

$$(2) \quad \mathcal{L}\left[f''\right] = s\mathcal{L}\left[f'\right] - f'(0)$$

$$= s\left[s\mathcal{L}(f) - f(0)\right] - f'(0)$$

$$= s^2\mathcal{L}(f) - sf(0) - f''(0)$$

and I get my second answer and so on I can go all the way up to the n^{th} derivative to get my answer.



Finally, I have one more example,

Example (12):

$$\text{Find } \mathcal{L}\left[t^n\right] \text{ by derivative formula .}$$

Solution:

$$f(t) = t^n$$

$$f^n(t) = n!$$

$$\mathcal{L}\left[f^n\right] = \mathcal{L}[n!] = n!\mathcal{L}[1] = \left(\frac{n!}{s}\right)$$

using derivative formula,

$$= s^n\mathcal{L}(f) - 0......$$

$$= s^n\mathcal{L}(f)$$

$$s^n\mathcal{L}(f) = \frac{n!}{s}$$

$$\mathcal{L}(f) = \frac{n!}{s^{n+1}}$$

So, from the derivative formula the answer is quite easy to see. Now, that completes our lecture today. In the next lecture I am going to talk about specific results related to Laplace transform namely the convolution theorem, the Parseval's inequality which are very important for us to use in some of the examples that we are going to solve using Laplace transform. Thank you very much.