Integral Transforms and Their Applications
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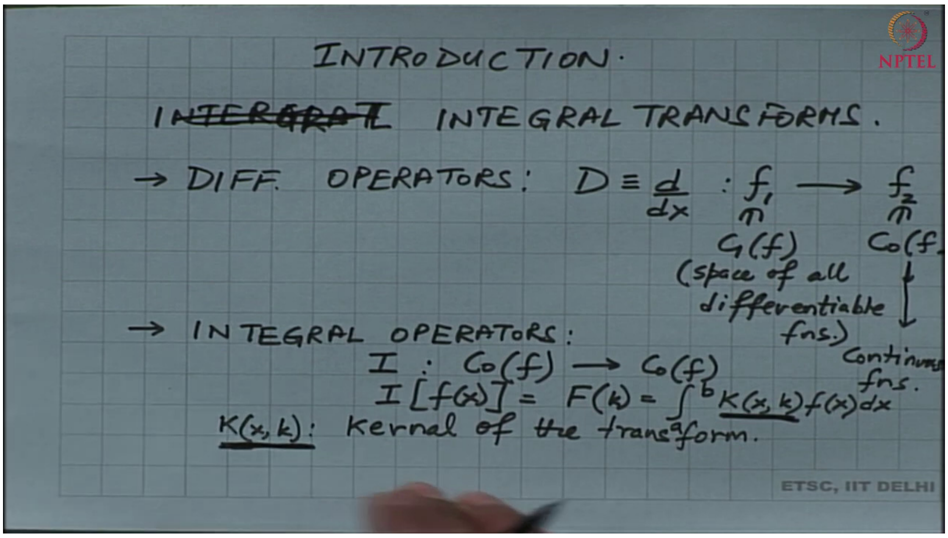
Lecture – 1
Introduction to Fourier Transforms Part - 01

Good morning everyone. I am Sarthok Sircar; and I will be teaching the course on Integral Transforms and their Applications. So, starting today it will be the first lecture. And in today's lecture, I am going to introduce the basic idea of integral transforms; followed by a brief definition and introduction on Fourier transforms, followed by its applications. So, without waiting much let us start the ideas of integral transform.



So, what is integral transforms? So, let me just introduce let me just introduce the idea of integral transforms. So, to understand the idea of integral transforms, we need to look at the differential operators first. So, what is or what do I mean by differential operators? the differential operators I denote it by .

$$D \equiv \frac{d}{dx}$$

when we operate this operator, it is operated on a function, so which means it is operated on a function f,which is coming from the space of c 1.

$$f_1 \longrightarrow f_2$$

$$C_1(f) \quad C_0(f)$$

So, what do I mean by c 1 for space, the space of all differential functions differentiable functions to all set of functions f. So, let me just let you know distinguish between this function and this function, where f 2 is also coming from the space this time of c 0 functions, which means the space of all continuous functions.

Now, similarly in the same vein of differential operators, let me just introduce the idea of integral operators. So, when I say integral operators I mean the following function

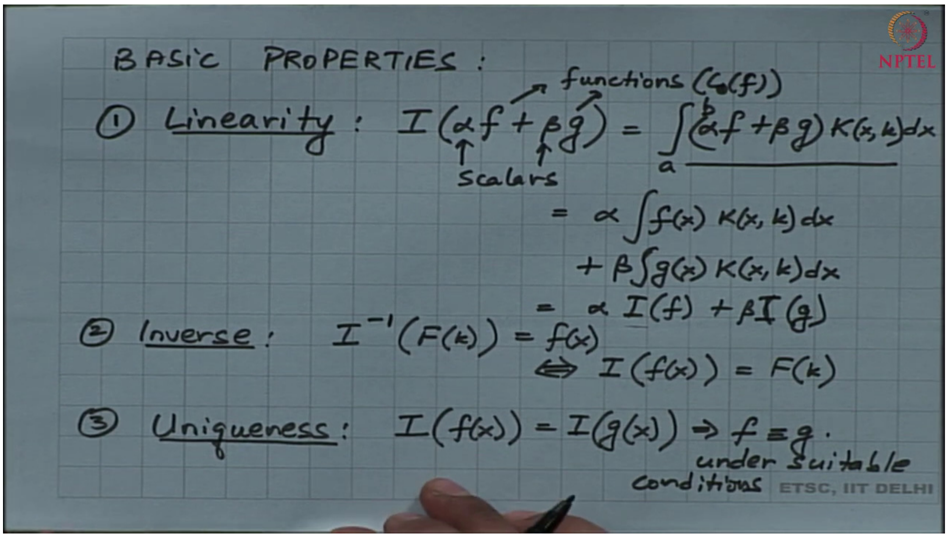
$$I_i : C_0(f) \to G(f)$$

let us denote it by some other notation let us say

$$I[f(x)] = F(k) = \int_a^b k(x,k)f(x)dx$$

So, since it is an integral operator, it is going to be an integration from some limit a to another limit b times a function of x and K that the transformed variable K and the physical variable x times f of x d x.

So, the function K we call it as the kernel, the kernel of the transform. And this kernel is going to vary from each definition of the integral transforms say starting from Fourier transform later onto Laplace transform, and many other transforms that I am going to introduce. So, it is the kernel function that changes in each of these integral operators. So, before I so let me just introduce few more properties of integral operators.



So, some of the basic properties, some of the basic properties of these integral operators.

1) Linearity: So the most basic property is that this operator is a linear operator. So, what do I mean by the operator being linear? So, suppose if I have this operator acting on the linear combination of two functions, where your these quantities alpha and beta these are scalars, and these are my functions coming from the space of c 1, so c 0, the set of all continuous functions. So, then this is going to be defined as integral from a to b alpha of f plus beta of g times the kernel function x comma k d x.

$$I(\alpha f + \beta g) = \int_a^b (\alpha f + \beta g)K(x,k)dx$$

So, then of course, we can write it this function as follows,

$$= \alpha \int f(x)K(x,k)dx + \beta \int g(x)K(x,k)dx$$

what conclusion have we found? We have found that the integral transforms of the linear combination of the functions is the linear combination of the corresponding integral transforms.

2) Inverse:The next property is that there is an inverse of this operator. So, I denote my inverse operator as

$$I^{-1}(F(k)) = f(x)$$

so I inverse of F of k is defined to be the original function if and only if I have I of f of x the function is capital F k given below:

$$I(f(x)) = F(k)$$

so that is the usual definition of inverse that we have. So, for each integral operator the inverse must exist.

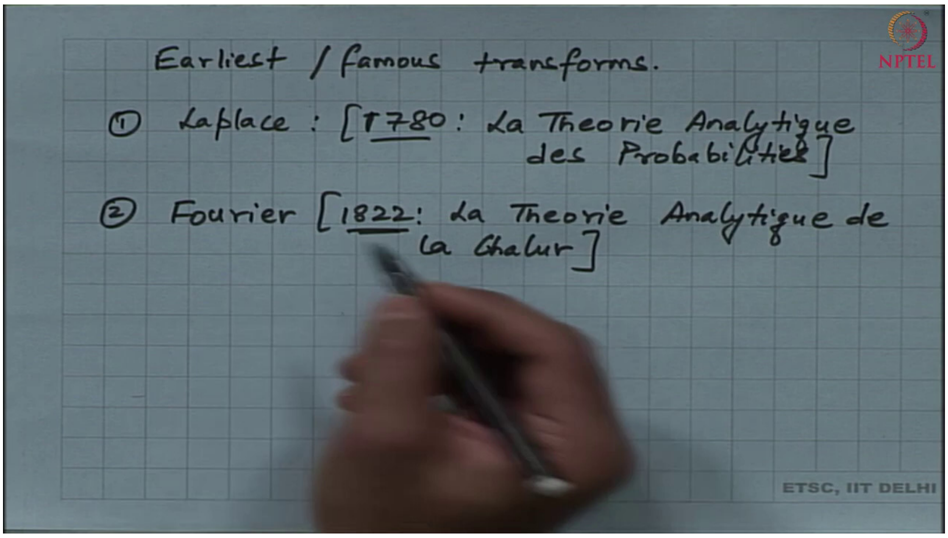
3) Uniqueness:The third property that is important to these operators are its uniqueness. So, when I talk about uniqueness, I am saying that

$$I(f(x)) = I(g(x))$$

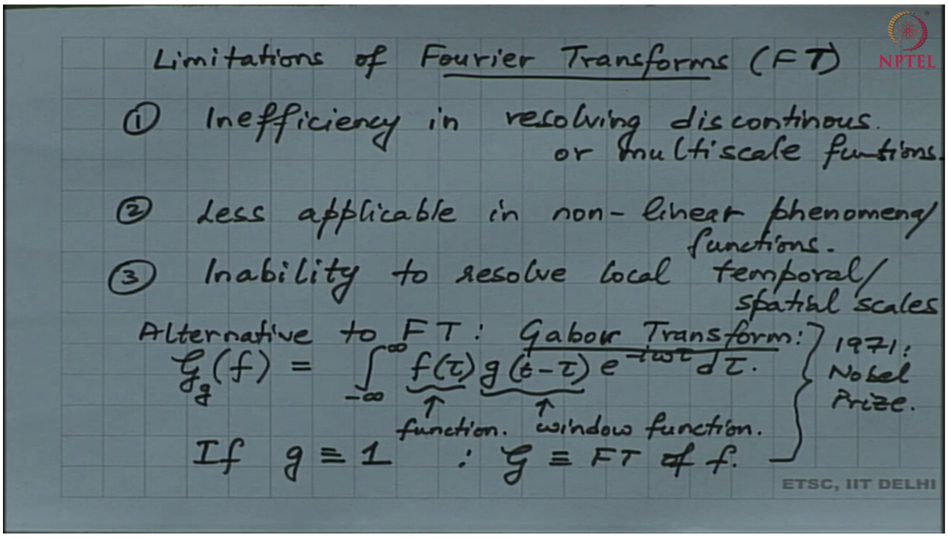
then it implies that the functions f is identically equal to g under suitable conditions,

$$f = g$$

So, we will see these suitable conditions as we move on from one case to another. So, then so these are some of the basic properties that these operators must have and then of course, when we move on to specific cases, we will see more properties of these transforms from Fourier to Laplace, and some of the other ones that we are going to study over our course of time. So, let me just you know provide you a little bit more motivation about these integral transforms.



So, integral transforms have been there for the last you know 250 years. And the first you know the first mention of these integral transforms were of course, by these famous mathematicians Laplace and Fourier. So, Laplace introduced the first mentioned of the transform was introduced by Laplace that was in 1780 in his book. And the name of the book if people want to look it up online the name of the book is La Theorie Analytique. So, it is a broke book in French La Theorie Analytique des probabilities. So, here was the first mentioned of Laplace transform. Then again another famous transform that was introduced was by Fourier, who introduced his famous Fourier transform in his first paper in 1822 again by the name of La Theorie Analytique De La Chaleur. So, it was a paper published in 1822. So, then of course so these are some of the earliest mention of integral transforms. Then you know of course, when we start looking at these transforms, although these transforms were very easy to define, there were some obvious limitations or some deficiencies of these transforms. So, some of these deficiencies that were mentioned and also noted by many of the researchers working in this area.



Limitations of Fourier Transform: So, let me just you know highlight some of the limitations of Fourier transforms. So, also let me for the ease of writing let me just denote my Fourier transforms as FT. So, from now on I am going to denote my Fourier transform by this shorthand notation FT.

So, some of the limitations were that they were not able to resolve, they were not able to resolve discontinuous or multi scale functions. So, I am going to show you some examples as to what do I mean by these limitations in one of our later lectures. The another limitation that was noted was that these are not applicable or I would say less applicable in non stationary or non-linear functions less applicable in non-linear.

So, the Fourier transforms of non-linear functions are not quite well defined as we will again sees in some of our examples later on. So, non-linear I would say phenomena or functions and we will see some examples in this case. The third limitation was that it is quite unable to resolve local temporal or spatial scales, temporal or spatial scales

So, what do I mean by that? That sub, So, what was shown here was that Fourier transform was able to locate the point in space where there is this sort of a spatial you know, it is able to resolve, it is able to tell you where there is a discontinuity. You know, but it is not able to pinpoint with accuracy at what point in space or what point in time this accuracy occurs. And again I am going to show you with some examples as to what, what do I mean by this problem.

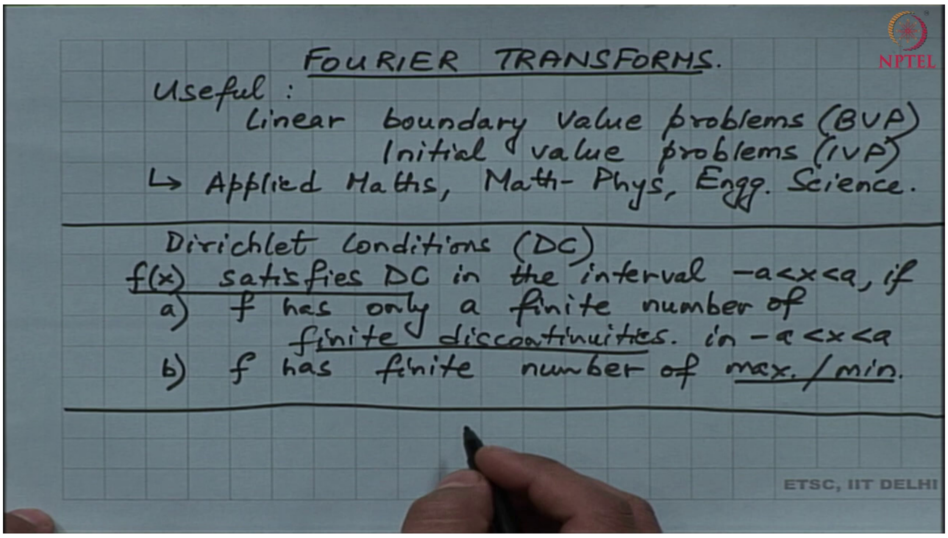
So, then of course, the moment you have limitations then people starts start to think about what are better ways to accurately represent our functions or phenomena with other transforms. So, then an alternative was proposed, an alternative was proposed to Fourier transform also known as the Gabor transforms.

So, just briefly I just want to mention, so what is this Gabor transform? Well the Gabor transform I am going to denote Gabor transform by this curly G. So, Gabor transform of a function f is to be defined as integral from minus infinity to infinity f of tau g t minus tau e to the power omega tau d tau. So, I call this as my, my argument. So, this is my function of which I am finding this transform. This is the function also known as the window function, the window function for the Gabor transform.

So, let, let me call this as g of g, where g is my this window function. Now, it is easy to see that if I have g is identically equal to 1 my Gabor transform is identically equal to the Fourier transform of f.

$$g_g(f) = \int_{-\infty}^{\infty} f(x)g(t-\tau)e^{-\iota\omega t}d\tau$$

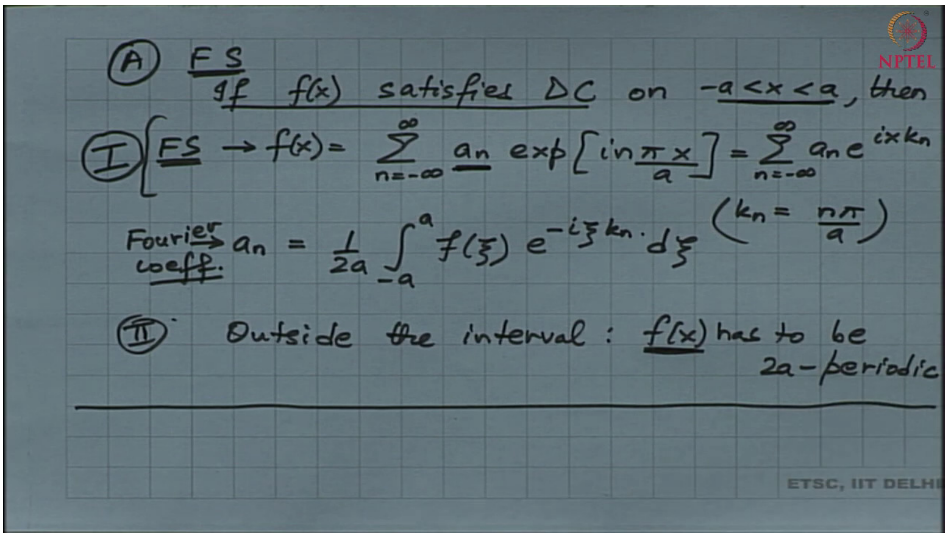
So, by the way this Gabor transform was introduced by a famous mathematician in 1971 in his Nobel Prize winning work. So, of course, his name is Dennis Gabor, and so this was one of the resolutions that was brought about to mitigate the limitations of Fourier transforms. So, let us now move onto more details in Fourier transform. So, specifically I am going to introduce the transform I am going to provide the definition. And of course, more relevant issues are I am going to talk about some examples and applications of Fourier transform.



So, before I move on let me just highlight the fact that Fourier transforms are quite useful, of course, in almost every aspect of science and engineering. So, starting from linear boundary value problems, and I am going to introduce all these problems later on. So, linear boundary value problems, I am going to denote it with this shorthand notation BVP, it is quite useful to solve problems in initial value problems or in shorthand notation I am going to call it as IVP. And as you can see that these problems arise everywhere starting from areas in applied maths to mathematical physics to engineering sciences, you name it and Fourier transform is already there. So, let me just start with the basic definition of Fourier transform. So, to start with the definition, I have to introduce one more notion that is the notion of Dirichlet condition. So, what do I mean by Dirichlet condition? So, let me just again denote it by the shorthand notation DC. So, if I have a function f, and f satisfies our Dirichlet condition in the interval from negative a to a, if I have the following,

1) f has only a finite number of finite discontinuities inside, this interval right.

2) the second condition that it must satisfy is that f has finite number of finite number of maxima or minima right. So, we are dealing with finite number of maxima or minima or finite number of discontinuities, then f is said to satisfy Dirichlet condition



Now, let me just introduce now I am going to introduce Fourier series. So, this will be introduced in our coming upcoming next module now. So, in this module, I am going to start with the definition of Fourier series and the definition of Fourier transform. So, Fourier series, so if I have my function f which satisfies Dirichlet condition on the interval from negative a to a, then f can be written as the following,

$$FS \to f(x) = \sum_{n=-\infty}^{\infty} a_n \exp\left[\frac{in\pi x}{a}\right] = \sum_{n=-\infty}^{\infty} a_n e^{ixk_n}$$

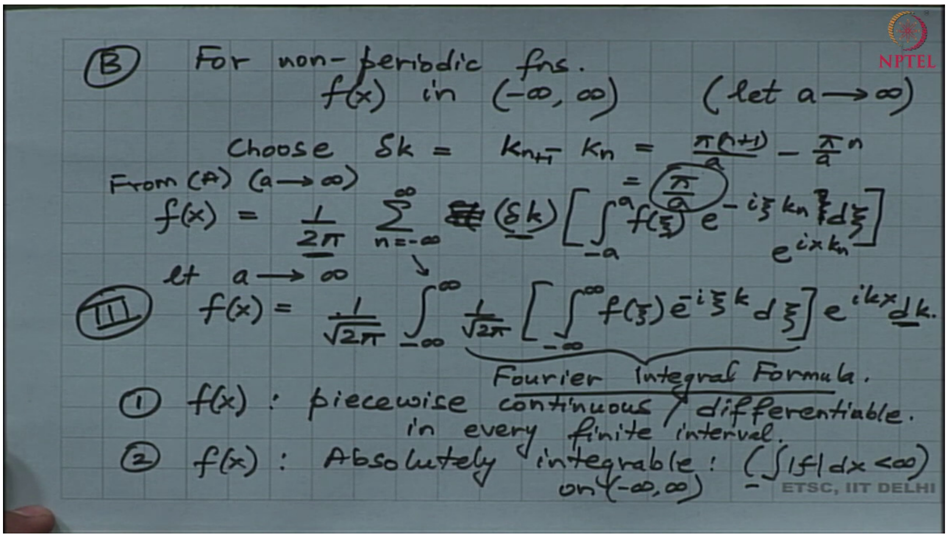
where my values kn:

$$k_n = \frac{n\pi}{a}$$

So, of course, we can always find these coefficients a n by our orthogonality condition,

$$a_n = \frac{1}{2a}\int_{-a}^{a} f(\xi)e^{-i\xi k_n}d\xi$$

So, you see that this is my Fourier series of the function f and I call these as my Fourier coefficients. So, a n s are my Fourier coefficients and f s is my Fourier series. So, one represents so let me just call this series and denote it by this number 1 or i. So, I is going to denote my Fourier series of course, you see that Fourier series are only applicable over an interval from minus a to a. So, suppose I want to use the concept of Fourier series, we must have the fact that outside the interval my function f has to be 2 a periodic. So, Fourier series are in general or necessarily must be for the case, where f x the function f is periodic.



Now in the case of non periodic functions for non-periodic functions, let us say f of x defined in the interval from negative infinity to infinity that is we are going to let our a in the earlier case, case a going to infinity right. Let me just also define another variable called delta k. where delta k is,

$$\delta k = k_{n+1} - k_n = \frac{\pi(n+1)}{a} - \frac{\pi n}{a} = \frac{\pi}{a}$$

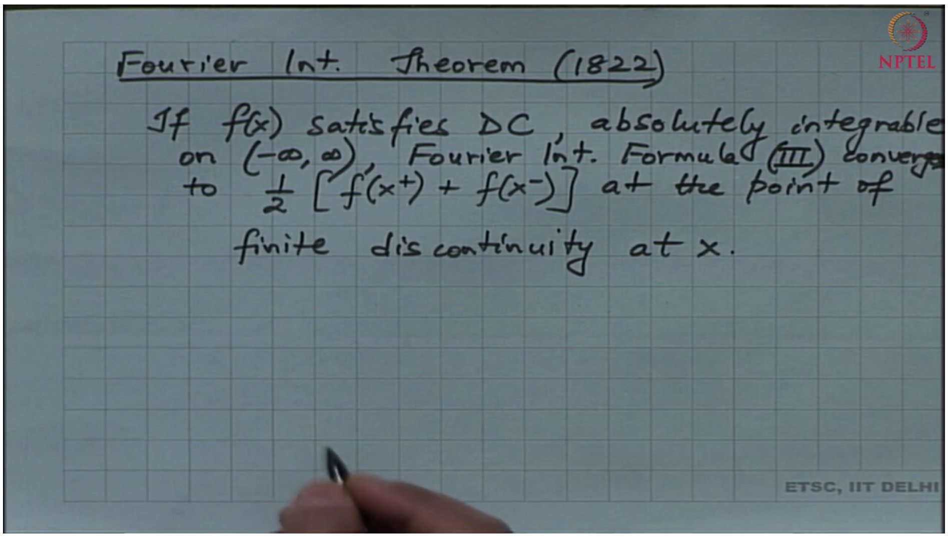
So, I have f of x, so again from a just go back and look at the expression for a, we are going to let a our interval going to infinity. So, f of x is given to be

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (\delta k) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(\xi) e^{-ik_n \xi} d\xi \right] e^{ik_n \pi}$$

Now, when we take limit a going to infinity this summation reduces to the following expression. So, let me just break down 2π as the follows as follows.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(x) e^{-i\xi k} d\xi \right] e^{ikx} dk$$

So, notice that our δ(k) has changed to this infinitesimal element dk and my summation has been replaced by an integral. And I have broken down this 1 by 2π as follows. So, I am going to denote this thing inside this integral as my Fourier transform or I call this as my Fourier integral formula, which is what I will be using for my Fourier transform. So, let me call this as my expression 3 for the Fourier integral formula. So, moving on before I do some application or examples, let me just highlight where Fourier transforms are you know can be evaluated. So, Fourier transforms are evaluated for all those functions, which are piecewise continuous, and not only piecewise continuous, but piecewise continuously differentiable right in every finite interval. And another requirement is that f(x) must be absolutely integrable. So, what do I mean by that? What I mean by that is, that this integral of absolute value of f must exist and is finite over any interval, let us say from minus infinity to infinity right, so absolutely integrable on minus infinity to infinity ok. So, let me just briefly state what is the Fourier integral theorem. I have already shown you what is the Fourier integral formula.



So, my Fourier integral theorem again published by for year n is 1822 paper is that if I have f(x), which satisfies my Dirichlet condition, and it is absolutely integrable on minus infinity to infinity. Then my Fourier integral formula Fourier integral formula, which was given in my expression 3 converges, it converges to this function $f(x^+) + f(x^-)$ at the point of finite discontinuity at x. So, I have already given you what is the Fourier integral formula. And it tells you that if you have a function with finite set of discontinuities and it is absolutely integrable, then the integral formula converges to this average value around the point of discontinuity.