Stochastic Processes
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Module - 7
Brownian Motion and its Applications
Lecture - 1
Definition and Properties

This is a stochastic processes module 7 Brownian Motion and it is properties, lecture 1 Definition and Properties. In the last 6 modules, we started with the review of probability is a one module, then the second module we discuss the definition of stochastic process and it is properties. And in the third module we have discuss the stationary process and it is all the properties. Fourth module we have discuss the discrete time Markov chain and in the fifth module, we have discuss the continuous time Markov chain, in the sixth module we have discuss the martingale, and this is the seventh module that is Brownian motion and it is properties.

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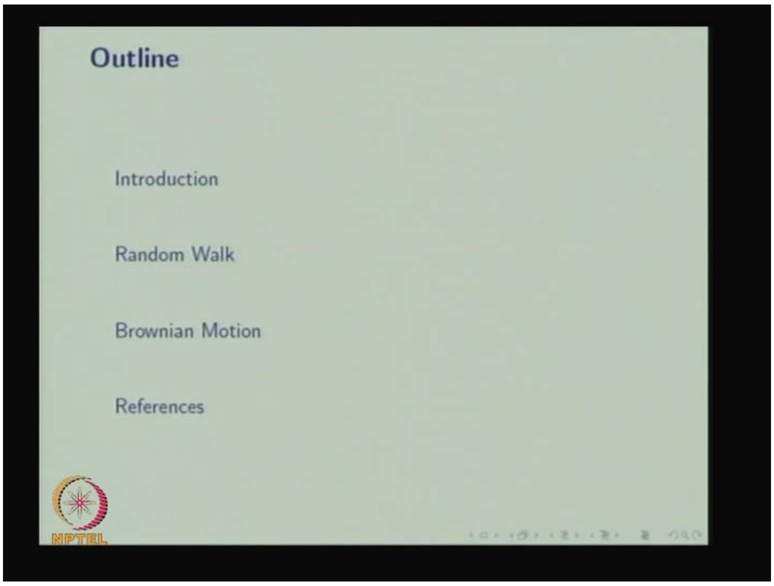

In this lecture, in this module we are planning to discuss the important stochastic process that is a Brownian motion, and then later we are going to discuss the process derived from the Brownian motion. Then we are going to discuss the stochastic calculus followed by that we are going to discuss stochastic differential equation and ito integrals.

And the application of the Brownian motion stochastic calculus, that is in the financial mathematics.

So, we are going to discuss the applications of Brownian motions as the in the financial mathematics, so with that the module 7 will be complete. And this is the lecture 1 of a this is the lecture one of a module 7 Brownian motion and it is application. In this lecture we are going to discuss the random walk and the definition of Brownian motion, then how one can derive the Brownian motion using a random walk; and some important properties of Brownian motions also will be discussed.

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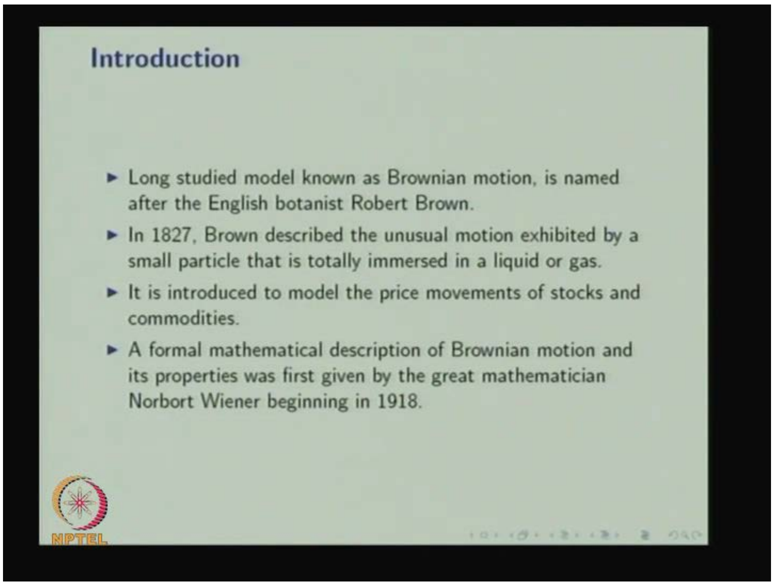

The long studied model known as a Brownian motion is named after the English botanist Robert Brown. In 1827, Brown described the unusual motion exhibited by a small particle, that is totally immersed in a liquid or gas. It is introduced to model the price movements of stocks and commodities. A formal mathematical description of Brownian motion, and it is properties was first given by the great mathematician Norbort Wiener beginning in a 1918. Therefore, the Brownian motion is also called it as Wiener process.

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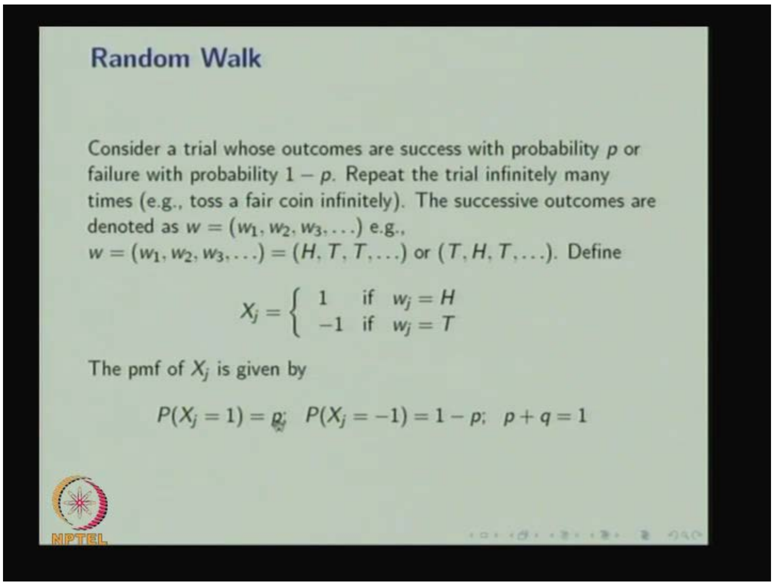

Now, we started with the random walk, because using this random walk we are going to derive the Brownian motion. Consider a trial whose outcomes are success with the probability p or failure with the probability 1 minus, repeat the trial infinitely many times that is equivalent of saying tossing a pair coin infinitely many times. The successive outcomes are denoted by the sample W that consists of a W 1, W 2, W 3 where each one is the outcome in the nth trial. That means, W 1 could be head or tail, similarly W 2 could be head or tail and so on.

For example, we have given H, T, T or T, H, T and so on; so, this collection is the this all the possible W's that is going to be the sample space. Now, we are defining the random variable X j it takes the value 1, if the outcome of the j th trial is head, if the outcome of the j th trial is tail, then the value is defined for X j is minus. So, this is the real valued function and this will be a random variable. Since it is takes a value 1or minus 1 this is the discrete random variable and one can find what is a probability mass function for the random variable X j.

So, since the trial whose outcomes are success with the probability p success is nothing but, heading a head and the failure is nothing but, trail land up with the tail. Therefore, the probability of X j is equal to 1 that probability is call it the w j is equal to head that call it as a success therefore, this probability is p. And the probability of X j is equal to minus 1 that is 1 minus p and this you can denoted by q, therefore p plus q is equal to 1; you can denote 1 minus p as a q, hence p plus q equal to 1.

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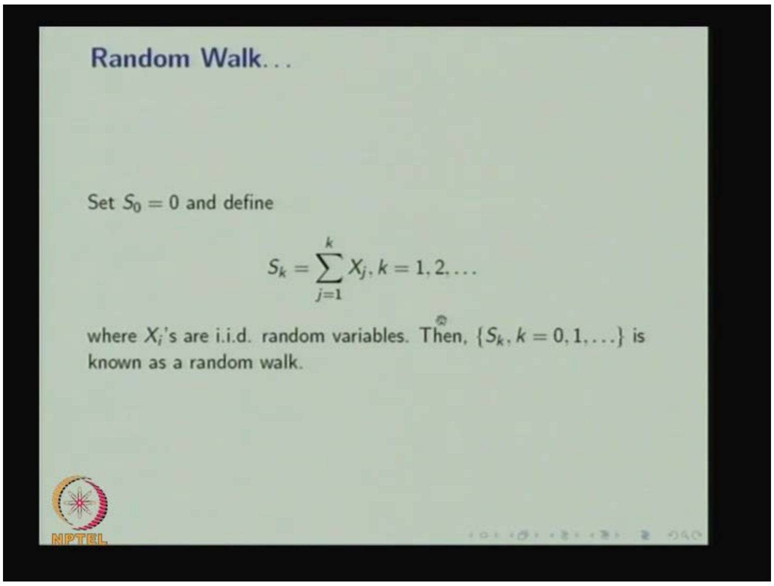

Now, we are defining the sequence of other random variables that is started with S naught is equal to 0, we are defining sum of first k random variables as a S k, where k is running from 1 2 and so on. Here X i's are i i d random variables and the sequence of random variables S k that is the random walk, with the S naught is equal to 0 and S k's are nothing but, the first k X i random variables.

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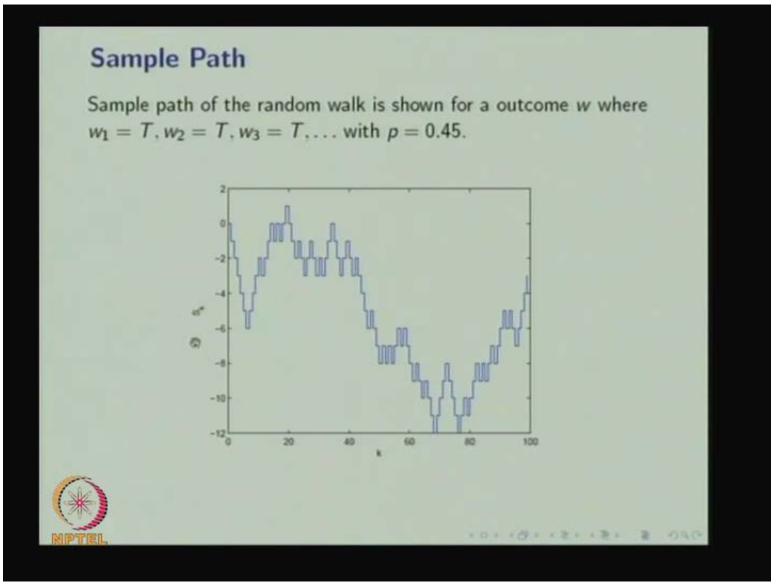

You can see the sample path of the random walk whose w one is a tail therefore, takes a value X 1 is minus 1 again if suppose w 2 is T, then X 2 also takes the value minus 1 suppose S 3 also T then X 3 also takes minus 1. Therefore, S k will be initially it is 0 then S 1 will be X 1 that is minus 1, S 2 will be X 1 plus X 2, that is minus 1 plus minus 1 that is minus 2. So, S 2 is minus 2, S 3 will be S 2 plus X 3 that is again adding minus 1, so S 3 will be minus 1, like that you can take the different values.

So, here this is a one sample path with w 1 is equal to T and w 2 is equal to T and w 3 is equal to T and so on; with the probability p is equal to 0.045 this is a probability of success or probability of a getting head when you toss a coin. We are going to conclude later as a tends to infinity using central limit theorem, one can conclude this will be a Brownian motion. For that, you should understand how the S k's are created where S k's are the sample path, where S k's are the random walk.

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Properties of Random Walk

- Choose non-negative integers $0 = k_0 < k_1 < \ldots < k_n$. Then

$$S_{k_{i+1}} - S_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j$$

Since X_j are i.i.d. random variables, $S_{k_1} - S_{k_0}, S_{k_2} - S_{k_1}, \ldots, S_{k_n} - S_{k_{n-1}}$ are mutually independent variables. Hence, $\{S_n, n = 0, 1, \ldots\}$ has the independent increment property.

- Similarly, for $0 \leq i \leq j$

$$S_j - S_i \equiv S_{j+h} - S_{k+h}$$

for $h \in N$ Hence, $\{S_n, n = 0, 1, \ldots\}$ has stationary increment property.

Now, we are going to see the properties of random walk if you choose a non negative integer's k naught k 1 and so on. Then if you find the difference, the difference is nothing but sum of X i's in this range. Since X i's are i i d random variables, if you take a non overlapping intervals are the increments of S i's then that will be mutually independent. Because, each this increments will be nothing but, the sum of few X i's and we know that each X i's are mutually independent i i d random variables.

Therefore, non-overlapping increments will be a mutually independent random variables. Hence S n's has the property called independent increment the increments are dependents independent. Similarly, for 0 less than or equal to i less than or equal to j S i minus S j is identically distributed with S j plus h minus S i plus h for h belonging to natural numbers.

Hence, the stochastic process S n has stationary increment property that means, if you find out the n dimensional random variable and shifted by h find out the another n dimensional random variable. If though that join distributions are same for both the n dimensional random variable without shifting and with shifting, then that stochastic process is called as stationary. But here, the stochastic process is not a stationary the increments are stationary means we have a increments and you shifted the increment by some interval h then the distributions are going to be identity. That is what it shows for one less than or equal to i less than j less than or equal to k less than l the difference the distributions are going to be same as long as the length is same. So, it is the increments are time invariance not the actual stochastic process, therefore this stochastic process has the stationary increment also; therefore, the random walk has increments are stationary as well as (( )).

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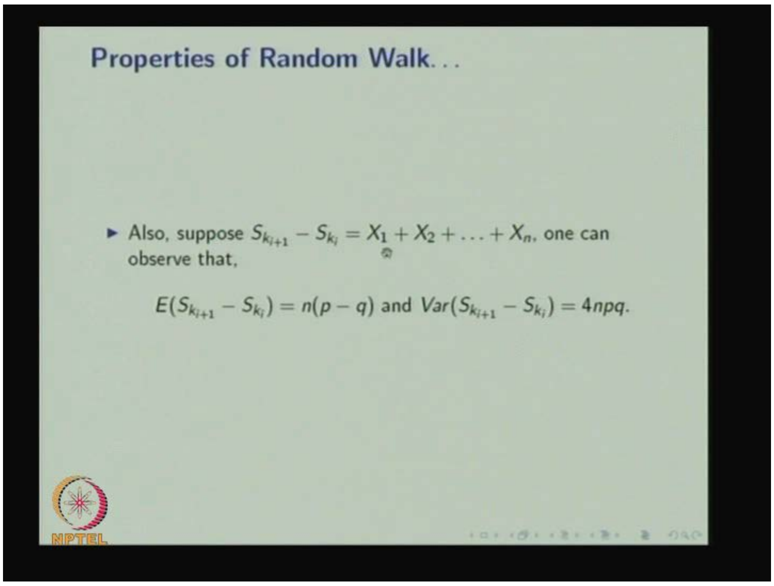

Also one can find mean and variance of increments, the increments are nothing but the difference of those random variables. And since each random variable are discrete type random variable with the probability mass function that is discussed in the previous slide. So, we can find out the mean and variance of those random variables, therefore we can find out the mean and variance of increments also. Now, we are going to derive the Brownian motion using random walk.

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Derivation

- Consider a particle performs a random walk such that in a small interval of time duration Δt, the displacement of the particle to the right or to the left is also of small magnitude Δx.
- Let $S(t)$ denote the total displacement of the particle in time t.
- Let X_j denote the length of the jth step taken by the particle in small interval of time Δt with pmf

$$P(X_j = \Delta x)=p;\ P(X_j = \Delta x)=1-p;\ p+q=1$$

$0<p<1$, where p is independent of x and t.

Consider, a particle performs a random walk such that in a small interval of time of duration delta t the displacement of the particle to the right or to the left is also a small magnitude delta x. Whenever a particle performs a random walk in a very small interval of time delta t, the displacement of a particle to the right or to the left that magnitude is delta x. Now, we are defining a random variable S of t denotes the total displacement of the (( )) any time t.

Let X j denote the length of the j th step taken by the particle in a small interval of time delta t with the probability mass function. So, the probability of the X j takes the displacement of the particle to the right side that is delta x with the probability p with the left side that is the X j takes the value minus delta x that is 1 minus p that is nothing but, a q where p plus q is equal to 1, where p is independent of x as well as time is very important.

The probability of the displacement to the right or to the left that probability, whether p or 1 minus p, which is independent of x as well as time. Now, the partition of the interval of length t into n equals sub intervals of delta x, then n times delta t becomes t and the total displacement S t is the sum of n i i d random variables X j, the varying partition the interval 0 to t into n equal parts.

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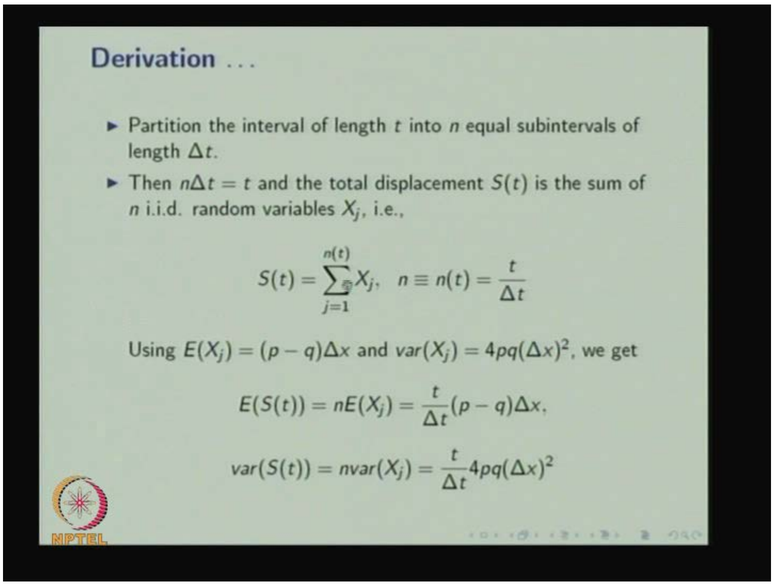

Therefore, the S of t the total displacement is nothing but, the sum of n i i d random variables X j's, where n is nothing but n of t, because you are partitioning the interval n you are partitioning the time interval 0 to t the length t into n parts. Therefore, n is nothing but n of function of that is nothing but t divided by delta t.

So, you know the mean and variance, therefore you can find out the mean and variance of S of t also because S t is a sum of n i i d random variables x j expectation is a linear operator. Therefore, n since it is a i i d random variable n times expectation of any one random variable. Whereas, variance since the random variables are independent, then the variance of S of t is nothing but variance of sum of random variables, so you can take it out and you can do the simplification.

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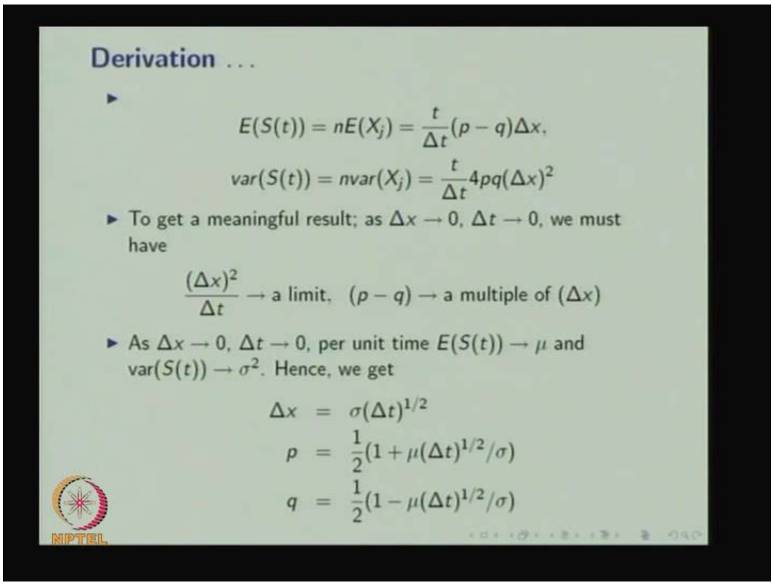

Now, you can make delta x tends to 0 as well as delta t tends to 0, therefore we will get a limit. By using the simple calculation the delta x is this much where p and q is equal to half times 1 plus mu times this and 1 minus mu times divided by delta.

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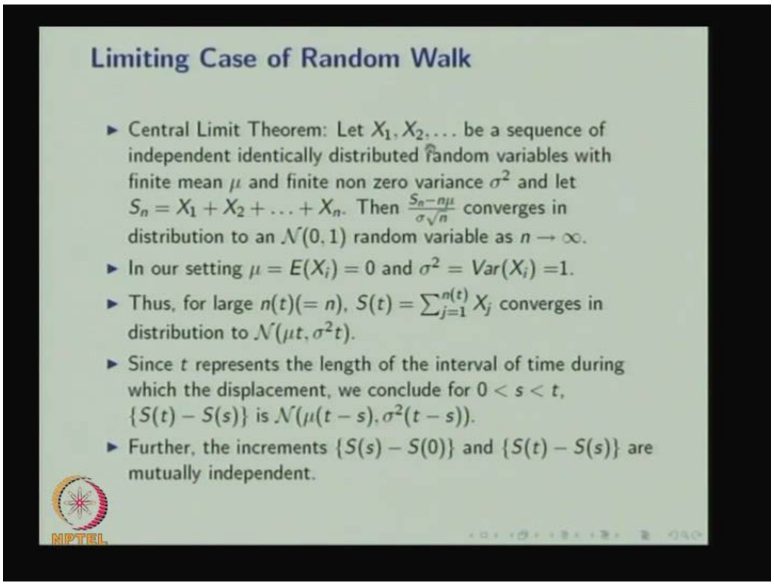

Now, we are using the central limit theorem let X 1, X 2 be a sequence of independent identically distributed random variables with the finite mean mu and finite nonzero variance sigma square. And let S n be a sum of 1 first n random variables, then S n minus the mean of this random variable divided by the standard deviation of this random variable converges in distribution to the normal distributed random variable with the mean 0 variance.

So, we are going to use this central limit theorem for our random walk scenario and for large n the n of t is equal to n where n is very large. Then conclude the S t converges in distribution to the mean of this random variable S of t that is mu times t and the variance of this random variable is sigma squared. Whereas, here we have use a central limit theorem the random variable minus their mean divided by the standard deviation converges to the standard walk.

But here, we are saying the S of t converges to the normally distributed random variable with the mean mu times t and the variance is sigma square t that is different from this mu and sigma square. Where mu is discussed here and the sigma is discussed here. Since t represents the length of the interval of time during which is the displacement therefore, instead of S of t you can go for S of t minus S of S. Since it is a t is a length of the interval.

Therefore, we can go for the S of t minus S of S that will converges to the normal distribution with the mean mu times t minus S and the variance sigma square times t minus S where S is less than. The way we discussed the properties of random walk it has the increments the, it has the property of increments or stationary as well as independent the same logic can be used here. So, here the increments S of S minus S of 0 are mutual independent increments are independent also.

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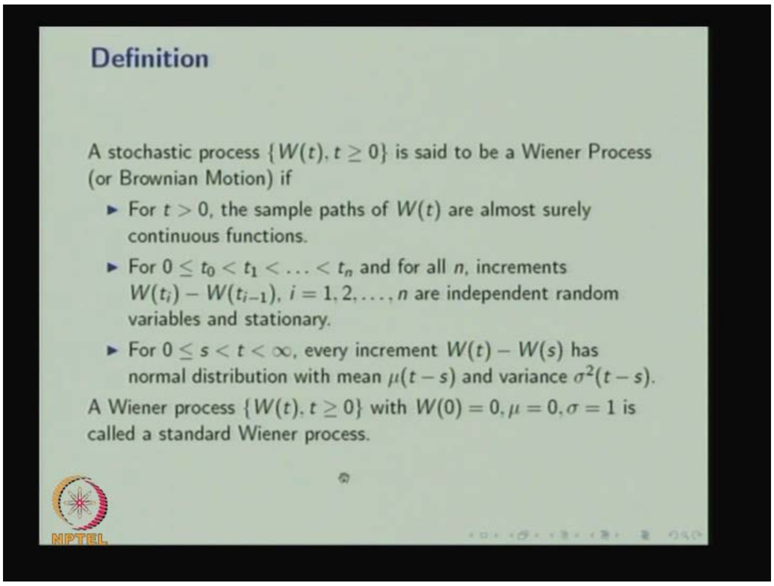

Now, we are defining the Brownian motion or wiener process. A stochastic process is said to be a wiener process or Brownian motion if it satisfies these three conditions, for t greater than 0 the sample paths of W t are almost surely continuous functions. For the interval 0 to t n in this form for all n the increments are independent as well as stationary the increments are independent random variables as well as stationary. And every increment has normal distribution with the mean mu times t minus S and variance sigma square t minus S this is what we have concluded.

In the limiting case of normal distribution the increments is normally distributed with the mean this much and the variance this much. So, this is what we have given as a conditions of a stochastic process will be a Wiener process. A stochastic a wiener process W t with mu of 0 is equal to 0 mu is equal to 0 and sigma square is equal to 1 is called as standard wiener process.

Whenever which is normally distributed with the mean 0 and the variance t minus s that means, the sigma square will be treated as one and the mu will be treated as 0 and also W 0 is equal to 0 then it is a standard Brownian motion or standard wiener process. So, any stochastic process satisfying these three conditions will be a wiener process or Brownian motion.

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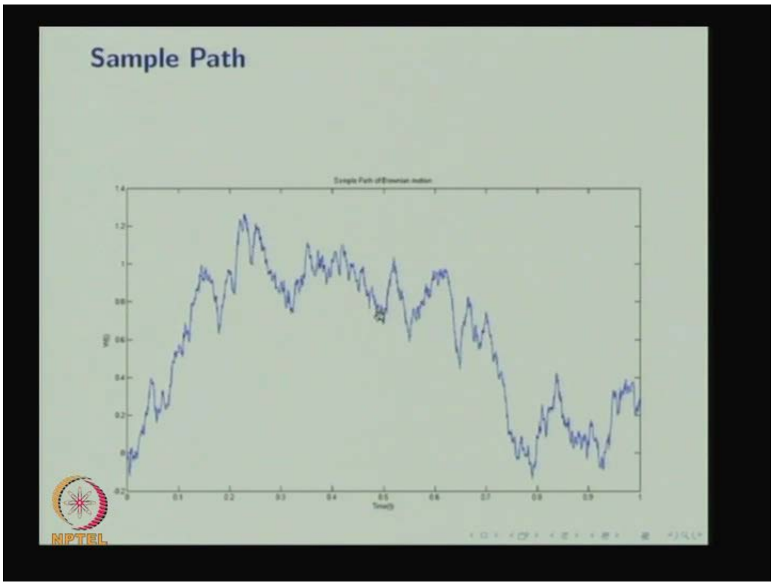

The sample path of wiener process it looks like this by definition W t plus S minus W t that is the increment follows normal distribution. It can take a positive and negative values the sample path of W t is a continuous, there is no jumps. And the limiting case of random walk will be the Brownian motion that also can one can visualize in the sample path of Brownian motion.

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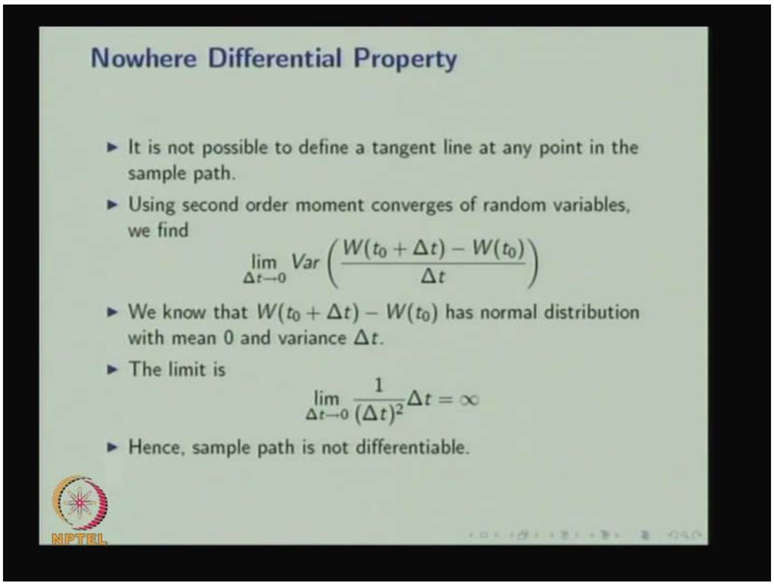

Now, we are going to discuss few important properties of Brownian motion. The first important property is nowhere differential you can see the sample path of Brownian motion, you see the sample path of the Brownian motion it is a continuous function but it has a too many fluctuation at every point. So, this is the one sample path. So, the first property says the sample path is not a differentiable anywhere or it is nowhere differential.

It is not possible to define a tangent line at any point in the sample you can see it in the sample path figure also. Using second order moment convergence of random variable we find the limit delta t tends to 0 the variance of the difference of this random variable divided by delta t. If we find out this limit if this limit is a finite then we can conclude it is differentiable at the point t naught.

Suppose you have real value function f of x and if you want to conclude the real value function f of x is (()) has a derivative at the point x naught. Then you should find out limit delta t tends to 0 f of t naught plus delta t minus f of t naught divided by delta t, if this limit is a finite, then you can conclude the real valued function f has the limit at t naught. Since the W's are the random variables and we know the mean and variance and also the distribution.

And the difference is going to be a random variable as a delta t tends to 0 it is going to be, we should find out the convergence of this difference of random variable divided by delta. So, one can use any mode of convergence to conclude to find out the limit delta t tends to 0 of this quantity. But here we are using the second order moment convergence, therefore we are finding limit delta t tends to 0 variance of this random variable.

The difference is a random variable difference divided by delta t is a random variable through we are finding what is a convergence of the function of random variable via second order moment of convergence. So, if you find out this quantity since this difference has normal distribution with the mean 0 and the variance delta t. Therefore, the variance delta t has to be treated as a constant.

So, the variance of 1 divided by constant time these will be 1 divided by delta t whole square and the variance of the difference of this random variable is delta t therefore, we will get infinity has delta t tends to 0. Since this limit is equal to infinity we conclude the sample path is not differentiable at t naught. Since t naught is arbitrary time point therefore, it is a nowhere differentiable or it is a, the sample path is not differentiable at every point every time.

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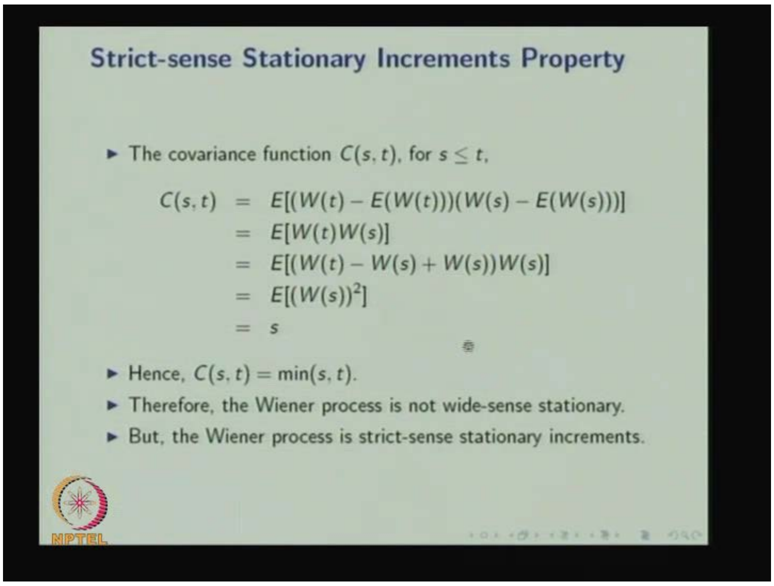

The second important property that is strict sense stationary increment, we are not saying the given stochastic process Brownian motion is a strict sense stationary. Here we are saying the increments are strict sense the increments are strict sense stationary, that means, the increments are satisfying the time invariant property. The strict sense stationary means, it has the time invariant property in the distribution. So, for that we are finding the covariance function you know the definition of covariance.

So, covariance of s comma t it is land up it is going to be S the covariance of S comma t is equal to minimum of s comma t, because here we have concluded for S is less than t it is s we make it as t is less than or equal to S we will get t. Hence c of s comma t is a minimum of S comma t, therefore wiener process is not wide sense stationary. Whereas, we can conclude it is a strict sense stationary increment, that means first we will find out the increments.

Then you one can prove for any finite dimensional the joint distribution is same as the joint distribution by shifting the time scale H. For every H the increments satisfying the condition the joint distribution are same the original joint distribution as well as the incremented by H. Therefore, it is going to be a strict sense stationary and the using the covariance function we are concluding it is not a wide sense stationary.

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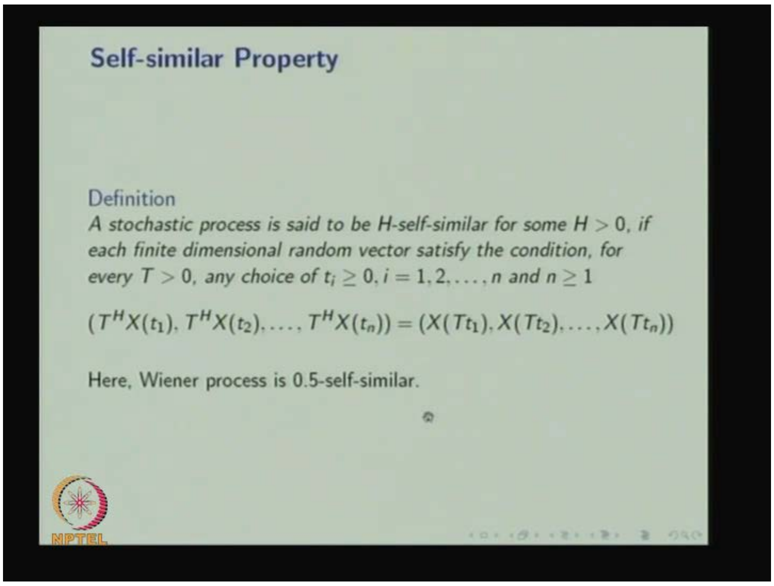

The next property is self similarity (( )). Let me give the definition of self similarity then we conclude wiener process is 1 by 2 self similar. It is a definition of self similar a stochastic process is said to be H self, similar for some H greater than 0. If each finite dimensional random vector satisfying the condition for every T greater than 0 any choice of t i's for i is equal to 1 to n, the joint distribution for n dimension random variable at the time points t 1, t 2 t n multiplied by T times H.

For every T and H is the H self similar for some H greater than 0 if that distribution is same as X of the time point is multiplied by T without H in the whole right hand side. So, if the joint distribution T times H 1 for the random variable X t times t 2 for the second random variable and so on. If this joint distribution is same as the joint distribution of this form then we say it is a H self similar for some H for every t greater than 0.

One can verify the wiener process is the 0.5 self similar. Here, I have not given the proof but you can multiply for some T for H is 0.5 you can conclude the wiener process is the 0.5 self similar.

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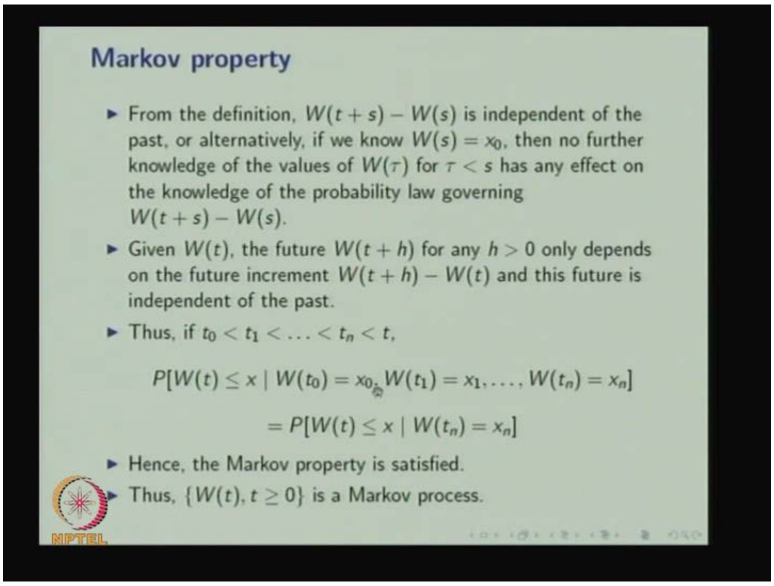

The next property, that is very important one that is Markov property, you know the definition of Markov property. So, this is the definition of Markov property, if any stochastic process satisfies the Markov property for arbitrary time point t naught to t n, which is less than t if this condition is satisfied, then the stochastic process will be a Markov process.

So, here from the definition one can conclude W of t plus s minus W of s independent of past or alternatively, if we know W s is equal to x naught, then no further knowledge of the value W of tau where tau is less than s has any effect on the knowledge of probability law governing W t plus S minus W s. The whole time scale the W of t plus s minus W s which is independent of the whole past history; and if we know the information at the s depends only at the time point S not the whole process from the definition you can make out.

Thus the definition says the increments are the increments are independent. Therefore, the W of t plus s minus W s is independent of the whole past information from 0 to s that is what it says. Therefore, given W t the future W of t plus h for any h greater than 0 only depends on the future increment W of t plus h minus W t and this future is independent of past.

Hence, this Markov property satisfied, since Markov property is satisfied for all the arbitrary time points t naught to t n therefore, this stochastic process is called Markov process. So, hence the Brownian motion is a Markov process.

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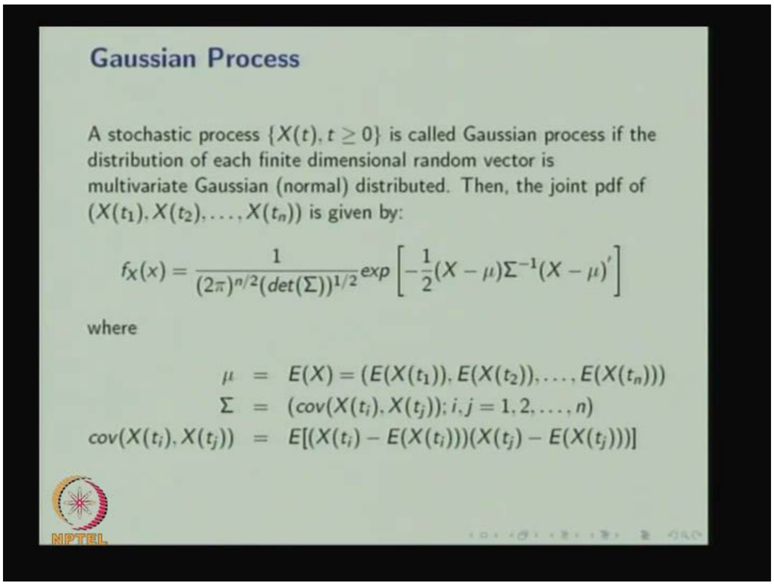

The next one is a Gaussian process first let me define what is Gaussian process then I am going to relate the Gaussian process with a Brownian motion. A stochastic process is called a Gaussian process if the distribution of each finite dimensional random vector is multivariate Gaussian distributed. That means, if you have a stochastic process and if you take a any finite dimensional random vector from that stochastic process, if that finite dimensional random vector is a multivariate Gaussian distributed random vector.

Then, the underlying stochastic process is a Gaussian process. Since for each finite dimensional random vector is a multivariate you can write down the joint probability density function of n dimensional random vector of Gaussian process. That is nothing but, this is the joint probability density function that is 1 divided by 2 power pi power n by 2 you find out the determinant of the matrix. And after that you find out the square root then exponential of where mu can be written as the vector and elements are nothing but the expectations.

And this notation sum is the covariance matrix covariance matrix covariance between any two random variables X of t i's with X of t j's where each one is running from 1 to n.

Therefore, it is the square matrix and elements are nothing but the covariance between any two random variables and all the diagonals will be the variance of X of t i's, where i is running from 1 to n. And it will be a symmetric matrix because covariance of X of t i comma X of t j is same as covariance of X of t j comma X of t i. Therefore, this matrix is a symmetric matrix and diagonal elements are variance of X of t i's. So, one can find out the covariance of any two random variable using this formula.

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Gaussian Process...

Since $\{W(t), t \geq 0\}$ is a Markov process as well as a Gaussian process,

$$P[W(t) \leq x \mid W(t_n) = x_n] \;=\; P[W(t) - W(t_n) \leq x - x_n]$$

$$= \int_{-\infty}^{x-x_n} \frac{1}{\sqrt{2\pi(t-t_n)}} exp\left[-\frac{s^2}{2(t-t_n)}\right] ds$$

Since W t is a Markov process as well as Gaussian process, you can write down the conditional c d. The conditional c d f is same as the difference (( )) less than or equal to x minus x n, but since this is normally distributed W t minus W of x n is a normally distributed. Therefore, this is nothing but minus infinity to x minus x n and this is the probability density function of normally distributed random variable with mean 0 and the variance t minus t n.

Whenever we whenever we will discuss the Brownian motion we are discussing a standard Brownian motion with W t is equal to W 0 is equal to 0 and mu is equal to 0 and sigma square is 1.

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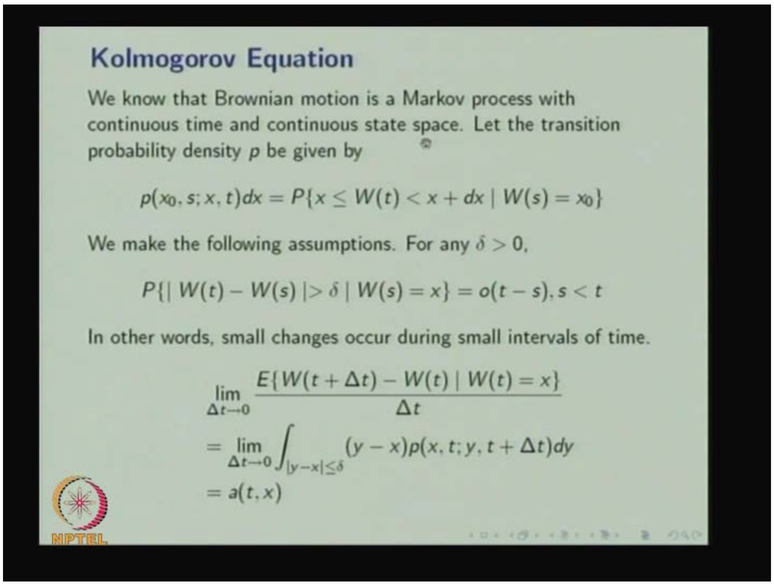

Now, we can discuss the Kolmogorov equation for the Brownian motion. We know that the Brownian motion is the Markov process with the continuous time and continuous state space, we can write down what is the transition probability density to the probability transition probability density p will be probability that W t is lies between x to x plus delta x t x given that W S is equal to x naught.

We make the following assumptions for any delta greater than 0 the probability of absolute W t minus W S which is greater than delta given that W S equal to x that is the order of t minus s. In other words the small changes occurs during small interval of intervals of time that is the meaning of the above equation. Now, we can find out the conditional expectation of W t plus delta t minus W t given W t is equal to x divided by delta t has limit delta t tends to 0 that is nothing but you can note down as the denoted as a of t comma x this will be a function of t comma x that is denoted as the a of t comma x.

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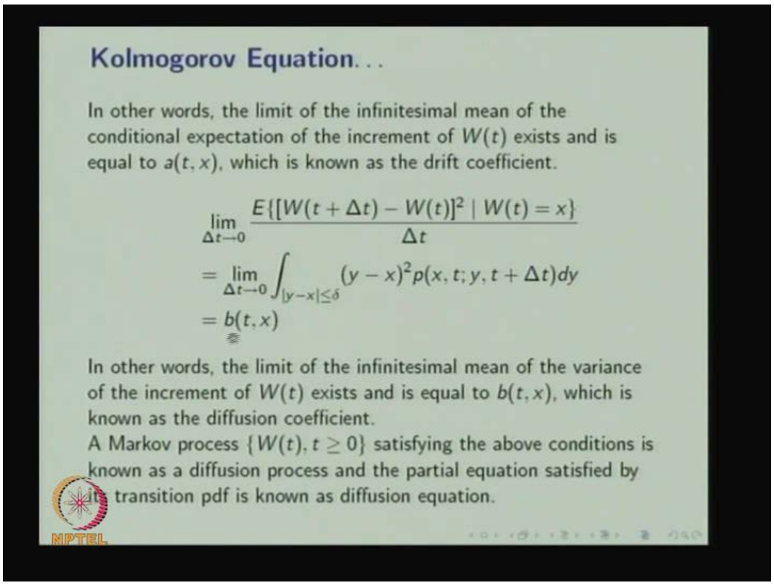

Similarly, we can make out the conditional expectation of the whole square given that W t is equal to x that you can denoted as the b of t comma x. In other words the limit of infinitesimal mean of variance of the increment W t exists and is equal to b of t minus which is known as the diffusion coefficient.

So, a Markov process W t is satisfying the above conditions is known as the diffusion process and the partial differential equation satisfied by its transition probability transition probability density function is known as a diffusion equation. The partial differential equation satisfied by it is transition probability density function is known as a diffusion function. So, this is the diffusion equation this is a p d e for the transition probability density function p and where a and b are earlier defined. This equation is also known as forward Kolmogorov equation and also known as a Fokker-Planck equation. And this equation is possible because of the W t is a Markov process. Therefore, and also it is a Gaussian process. Therefore, we will land up the transition probability density function p and satisfying the p d e and this p d e is called the Fokker -Planck equation.

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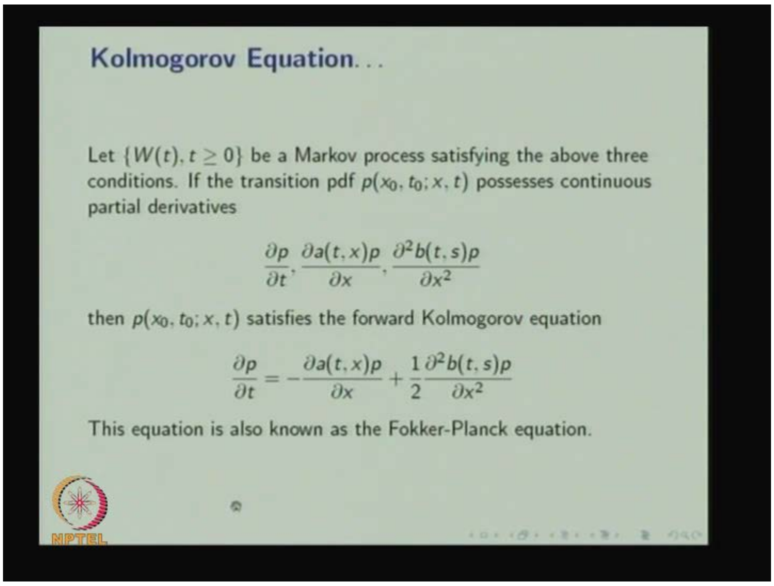

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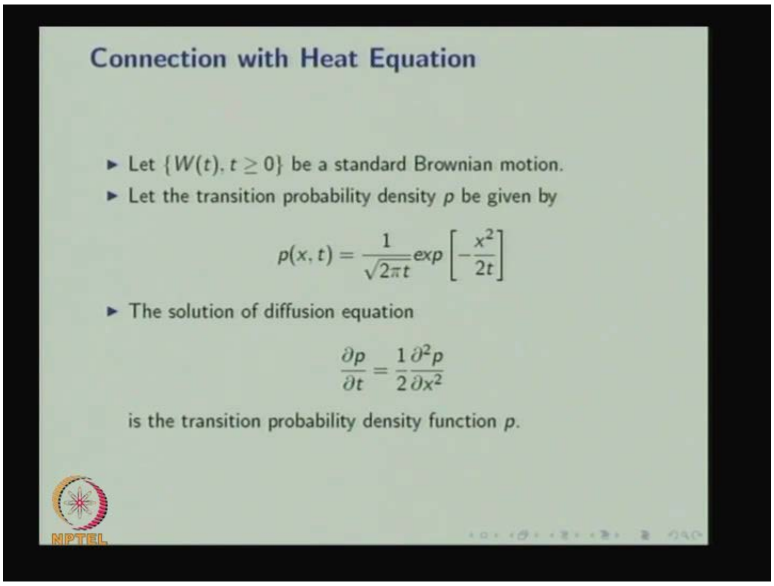

If you solve the p d e which is given here for the standard Brownian motion or the standard means W 0 is equal to 0 mu is equal to 0 and sigma square is 1 in the definition of Brownian motion. Then, you will get the transition probability density function p is 1 divided by square root of 2 times of pi t exponential of minus x square by 2 times t. And this is the probability density function of standard normal distributed random variable with mean 0 and the variance t. And the corresponding diffusion equation is dou p by dou t is equal to 1 by 2 dou square p by dou x square.

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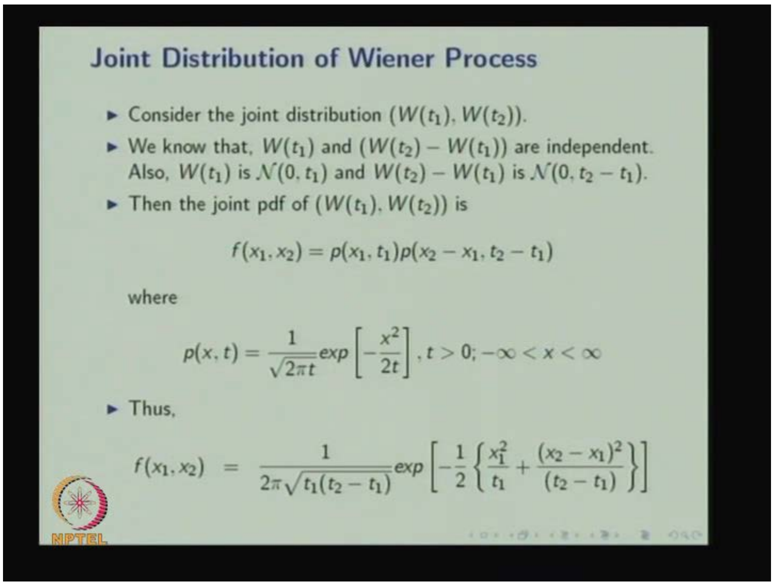

Now, we are going to discuss the joint distribution of wiener process the way we discuss the Gaussian process because the Gaussian process every finite dimensional random vector is a multivariate random multivariate normally distributed random variable. Therefore, you can find out the joint distribution of W of t 1 with W of t 2, we know that W of t 1 and W of t 2 minus W of t 1 are independent here we made t 1 is less than t. And also we know that W of t 1 is normally distributed with the mean 0 variance t 1 and this difference is also normally distributed with the mean 0 and variance t 2 minus t 1 and both are independent.

Our interest is to find out the joint distribution of W of t 1 with W of t 2 but for that first we find out the joint distribution of W of t 1 with W of t 2 minus W of t 1, then use a function of a function of a random variables 2, then you can find out the joint distribution of these two.

So, first we, so that is the way here I have not given the derivation. So, finally you will get the joint distribution of joint probability density function of W of t 1 with W of t 2 is in this form where the probability density function is going to be the normally distributed random variable. Hence the joint distribution will be one divided by square root of 1 divided by 2 pi times square root of t 1 times t 2 minus t 1 exponential of this expression.

Note that note that W of t 1 and W of t 2 are not independent, whereas W of t 1 with W of t 2 minus W of t 1 are independent random variables. So, using that we are finding the joint distribution of W of t 1 with W of t.

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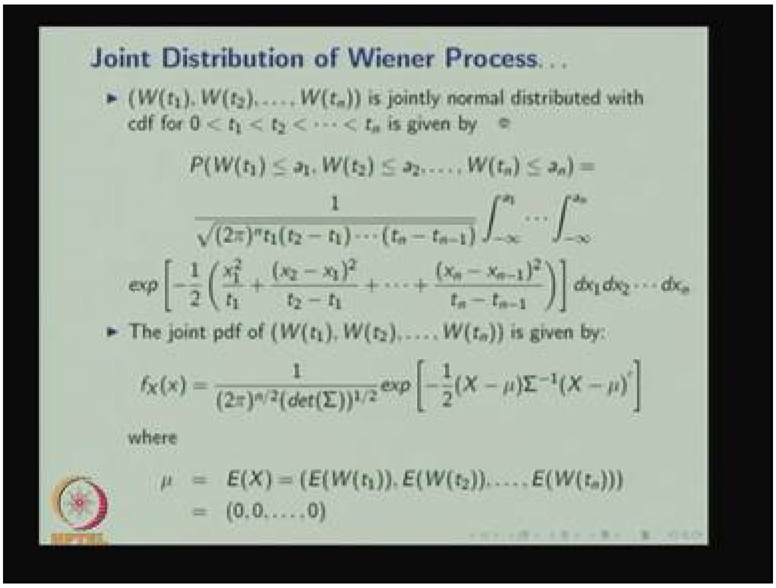

Once you know the joint distribution for any two random variables the same way you can find out the joint distribution of any n random variables in the wiener process also in the same way. I am not given the derivation here and you can find out the joint distribution joint probability density function of n random variables also.

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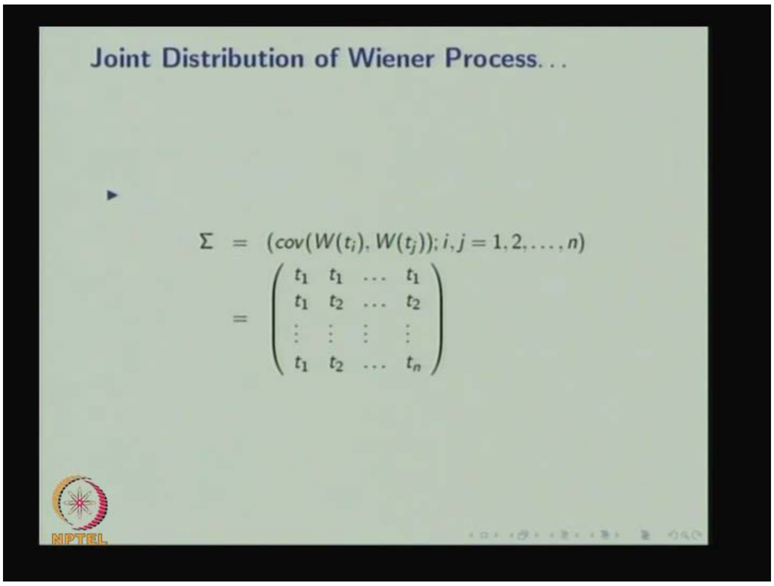

And we need a covariance matrix and expectation, so the expectation vector that is mean therefore, all the means are 0. Whereas, the covariance already we got the covariance of any two random variables of W of t 1 with W of W of t i's with W of t j's it will be a symmetric matrix and diagonals are nothing but the variance of W (( )).

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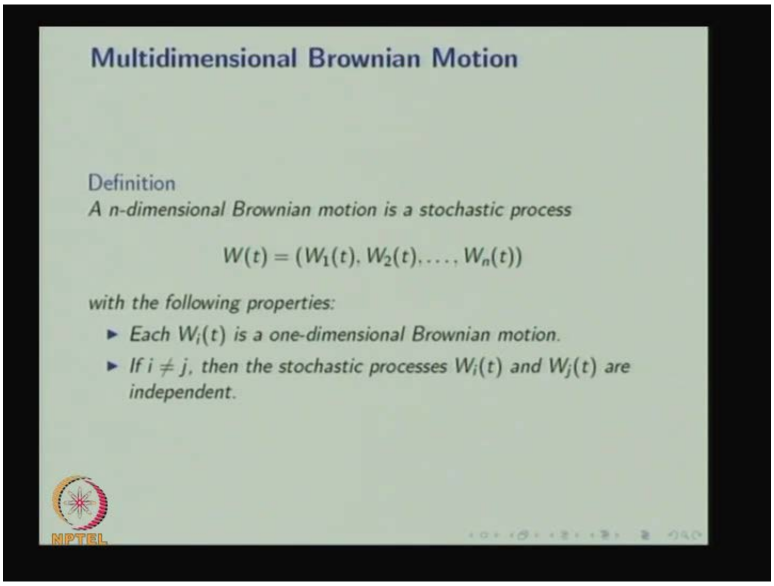

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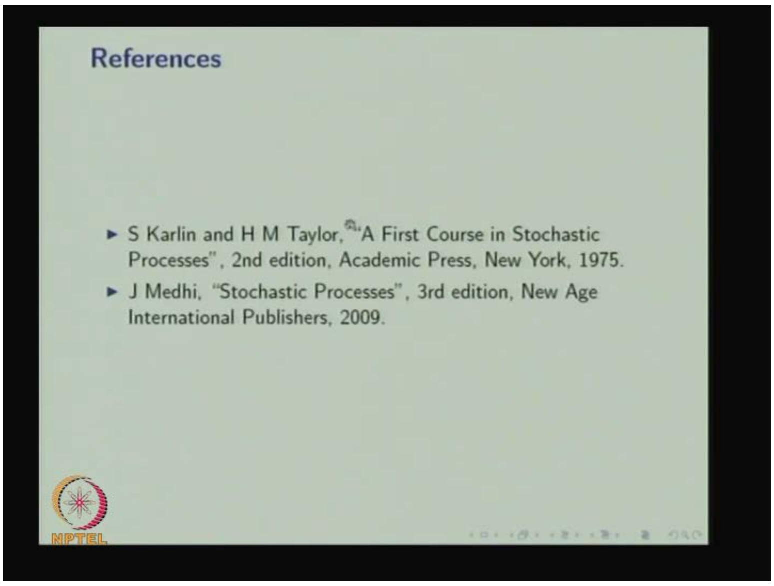

We can go for the multi-dimensional Brownian motion, we can have W 1 is a Brownian motion, W 2 is another Brownian motion so we can collect it as make it as another W t and each W is a one-dimensional Brownian motion. And you can go for the stochastic process are independent, therefore we will have n-dimensional Brownian motion also.

Here is the reference, so in this lecture we have discuss the definition of Brownian motion and also we discussed the derivation of Brownian motion. And we have discussed important properties of Brownian motion starting from stationary increment increments are independent Markov property, martingale property. And also finally, we discuss the multidimensional Brownian motion.