Point Set Topology
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Week 02
Lecture 08

Okay, so welcome to this lecture. In the previous lecture we defined continuous maps. We had ended the lecture by this lemma, which I had left as an exercise. So, let us just prove this lemma, perhaps some of you are able to do it. Let X and Y be topological spaces. Let f: X -> Y be a map of sets, and we want to check that F is continuous.

Let $\mathcal{B}$ be a basis for the topology on Y. Then if f^{-1}(V) is open for every V in $\mathcal{B}$, then f is continuous. This lemma gives us a convenient criterion to check whether a function is continuous in terms of checking only for basic open sets. So, let U ⊆ Y be an open set.

Then for each x in U there exists a basic open set B_x, such that x belongs to B_x and B_x ⊆ U. Therefore, this shows that we can write U as a union over all the x's, of these B_x's. Then it is a straightforward check in set theory that f^{-1}(U) is equal to union of all f^{-1}(B_x)'s. Each of these is open, as this B_x belongs to $\mathcal{B}$ and by hypothesis, the assumption is that f^{-1}(B_x) is open, and as arbitrary union of open sets is open, this implies f^{-1}(U) is open. And since this happens for every open set U, thus f is continuous.

Next we want to show that the space for continuous functions which are real valued or complex valued maps has some nice algebraic structures. So, to do that we are going to begin with the following theorem. So, before that consider ℝ and ℝ^2 with the standard topology, and there are two natural maps which we can define. So, then the following maps. So, the first one is the addition map.

This is given by a(x,y) = x+y, and the second is the multiplication map, this is given by m(x,y) = xy, is the multiplication. So, both these are continuous. So, we will use the above lemma to prove this theorem. So, it suffices to show that a^{-1}(B_ε(z)), let us say z is a real number and ε > 0, on the standard topology on ℝ has these as basic open sets, and we have to show that these the inverse images of these are open. So, in order to do that.

So, let (x,y) be in ℝ^2, and recall that we had defined S_ε(x,y) to be those (x',y') in ℝ^2 such that |x' - x| < ε and |y' - y| < ε. So, note that if (x', y') is in this set, then |(x' + y') - (x + y) | <= |x - x'| + |y - y'| <2ε This implies that |a(x', y') - a(x,y)| < ε for (x', y') in S_ε(x,y). So, this shows that S_{ε/2}(x, y) is contained in a^{-1}(B_ε(x+y)) Let us check why this happens. To see this, we need to show that if (x', y') is over here, then a(x', y') belongs to

$B_\varepsilon(x+y)$, but we have checked that if (x', y') is here then $|a(x', y') - a(x,y)| < 2(\varepsilon/2) = \varepsilon$ but this $a(x, y)$ is precisely $x+y$. This implies that $a(x', y')$ belongs to $B_\varepsilon(x+y)$.

So, we have simply rephrased the above computation in this statement. Let us go ahead. Let, now using this, we will prove that in the inverse image of f, all basic open sets are open. So $B \subseteq \mathbb{R}$ be a basic open subset. Suppose (x, y) is in $a^{-1}(B_\varepsilon(z))$.

This implies that $x+y$ belongs to $B_\varepsilon(z)$. This $B_\varepsilon(z)$ is the epsilon ball around z and $x+y$ is somewhere over here. So, this implies we can find $\varepsilon' > 0$ such that this $B_{\varepsilon'}(x+y)$ is completely contained in $B_\varepsilon(z)$, and from the above computation, it follows that $S_{\varepsilon/2}(x,y)$ is contained in $a^{-1}(B_\varepsilon(x+y))$ which is contained in $a^{-1}(B_\varepsilon(z))$. I should write $S_{\varepsilon/2}(x,y)$. What this shows is that, thus for every (x, y) in $a^{-1}(B_\varepsilon(z))$, there exists some $\varepsilon' > 0$ such that $S_{\varepsilon'/2}(x,y)$ is contained in $a^{-1}(B_\varepsilon(z))$.

So, therefore by the definition of the standard topology on $\mathbb{R}^2$, $a^{-1}(B_\varepsilon(z))$ is open in $\mathbb{R}^2$. This shows that the addition map is continuous. Similarly let us show that the multiplication map is continuous. We prove the second part now. Once again, we start with a small computation.

Let us start with a $0 < \delta < 1$, and suppose that (x', y') belongs to $S_\delta(x, y)$. This is in $\mathbb{R}^2$. Then, $|x'y' - xy| = |x'y' -xy' +xy' -xy|$. $\leq | x'- x || y' | + |x||y' - y|$, which is strictly less than $|x' - x|(|y| + \delta)$ (This is because $|y' - y|<\delta$. This implies that $|y'| < |y| + \delta) + |x| |y' - y|$.

So, this is strictly less than, Now this is also less than δ, this quantity over here. So, we use that and so this quantity is less than δ, plus $\delta(|y| + |x| + \delta)$, which is strictly less than $\delta(|y| + |x| + 1)$. This computation shows that, we can just rephrase this computation as, when we apply m on $S_\delta(x,y)$, this is going to be contained in $B_{\delta (|x| + |y| + 1)}(xy)$ So, once again, let us check why this shows this. This is because if (x', y') is in this set $S_\delta(x, y)$, then $m(x', y') = x'y'$, we have just now checked that over here this belongs to $B_{\delta(|y|+|x|+1)}(xy)$. So, sorry there is a typo over here, So, now let us fix any $0<\varepsilon<1$ and define $\delta := \varepsilon/(|x|+|y|+1)$.

So, then clearly $0 < \delta < 1$. So, as a result what happens is this implies that $m(S_{\varepsilon/(|x|+|y|+1)}(x, y)) \subseteq B_\varepsilon(xy)$, which is the same as saying that $S_{\varepsilon/(|x|+|y|+1)}(x, y)$ is contained in $m^{-1}(B_\varepsilon(xy))$. So, now with this computation in mind, suppose (x, y) belongs to $m^{-1}(B_\varepsilon(z))$, where z belongs to $\mathbb{R}$, and $\varepsilon>0$. Then there is an $0 < \varepsilon' < 1$. So, this is our z, and this is ε, and xy is somewhere over here.

So, we can always choose ε' and further assume that $0 < \varepsilon' < 1$, such that this ball of radius ε' around xy is contained in this ball of radius ε around z. So, from the above, we conclude that $S_{\varepsilon/(|x|+|y|+1)}(x, y)$ is contained in $m^{-1}(B_\varepsilon(xy))$, which is contained in $m^{-1}(B_\varepsilon(z))$. So, once again, therefore given any point (x,y) in $m^{-1}(B_\varepsilon(z))$, we have found this positive quantity, such that this basic open set around xy is completely contained in $m^{-1}(B_\varepsilon(z))$. This implies that $m^{-1}(B_\varepsilon(z))$ is open. This shows that the multiplication map is also continuous.

So, this completes the proof of the theorem. So next we prove a similar theorem, but about $\mathbb{R}\setminus\{0\}$. So, so let $\mathbb{R}^*$ be the set, we remove 0 from the real line, with the subspace topology. And consider, there is a very nice map on this set. So, consider the map f, there is a natural map given by $f(x) = 1/x$, and the claim is this map is continuous.

So, let us prove this claim once again. So, let us begin with an x in $\mathbb{R}^*$. So, one checks easily that if $\varepsilon < |x|$, then the this basic open set is completely contained in $\mathbb{R}^*$, that is obvious, because this is, let us say this is 0. So this is $\mathbb{R}^*$, we have removed 0, and if you take any x and we choose an $\varepsilon < |x|$, then this neighborhood, this interval around x will be completely contained in $\mathbb{R}^*$ and the thing is, it is easy to check that the collection $\{B_\varepsilon(x) : x \text{ in } \mathbb{R}^* \text{ and } \varepsilon< |x|\}$ is a basis for the subspace topology on $\mathbb{R}^*$. Therefore, we will check that.

Thus it suffices to check that $f^{-1}(B_\varepsilon(x))$ is open when $\varepsilon < |x|$. So, in this situation, what do we have? We have removed 0 and let us say our x is somewhere over here. We take this neighborhood $(x-\varepsilon, x+\varepsilon)$, then one easily checks that this set $f^{-1}((x-\varepsilon, x+\varepsilon))$ is precisely equal to the interval $(1/(x+\varepsilon), 1/(x-\varepsilon))$, which is open in $\mathbb{R}^*$ in the subspace topology. This proves that f is continuous. So, we will end this lecture here.