Point Set Topology
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Week 02
Lecture 06

Let us recall that in the previous lecture Given topological spaces X and Y with topologies, let us say $\mathcal{T}$_X and $\mathcal{T}$_Y, we define a topology on X × Y. So, how do we do this? We first define the collection $\mathcal{B}$, a subset of P(X × Y) and it is defined as follows: it is a set of U × V, such that U is open the topology in X and V is open in the topology in Y. And we check that $\mathcal{B}$ satisfies the two conditions required to generate a topology, namely (a): when we take the union over all subsets W in $\mathcal{B}$, we should get the entire space. In this case it is X×Y, and the second is, given W_1 and W_2 in $\mathcal{B}$ and a point x in the intersection, then there exists a W in $\mathcal{B}$ such that x is in W and W is contained in W_1∩W_2. So, we check that $\mathcal{B}$ satisfies these two conditions, and using these, we define the topology on X×Y with $\mathcal{B}$ as basis. Let us observe that the same idea can be used to put a topology, first on a finite product i = 1,2, .

.., n of X_i's Here the X_i's are topological spaces with topologies $\mathcal{T}$_(X_i)'s We define $\mathcal{B}$ ⊆ P(X_1 × X_2 × .

.. × X_n) as $\mathcal{B}$ is equal to product of U_i's, where each U_i is in $\mathcal{T}$_(X_i), and I will leave this as an exercise. It can be done in the same way. So, we have already done this when n = 2, and the same proof will show that $\mathcal{B}$ satisfies the two conditions required to generate a topology on this product. So, this topology that is the topology generated by $\mathcal{B}$ is called the product topology.

Now we took a finite product of topological spaces, we can ask what happens in the case of a infinite product. Let us consider the case: Let I be a set (possibly infinite) and assume that for each i in I we are given topological spaces for each i in I we are given a topological space X_i. So, how can we put topology on the set product (on all i in I) of X_i's. So, now here there are two possible candidates for topologies here. We can define $\mathcal{B}$ in two ways.

In the first case, we can define the collection $\mathcal{B}$ to be the most naive one, set product from i in I of U_i's, where each U_i is in $\mathcal{T}$_(X_i). This is the first candidate. It is easy to check again that, let us call this $\mathcal{B}$_1, this collection $\mathcal{B}$_1 satisfies the two conditions required to generate topology and so this defines topology let us say $\mathcal{T}$_1 on this product. This topology is called the box topology. The other candidate for $\mathcal{B}$ is as follows: let us call this $\mathcal{B}$_2.

So, this is product of i in I of U_i's such that, we put the extra condition that $U_i = X_i$ for all but finitely many i's. So U_i belongs to $\mathcal{T}_{(X_i)}$, this is the first condition, let us call this (a). The second condition we want is, when we look at the collection of i in I such that U_i is not equal to X_i, recall that each $U_i \subseteq X_i$. So we look at the collection of those indices for which U_i is a proper subset of X_i and what we want is (that set) should be finite. This collection over here that should be finite, which means that except for finitely many indices, all the U_i's $=X_i$.

This topology, so once again check that $\mathcal{B}_2$ satisfies the two conditions to define a topology $\mathcal{T}_2$, and this topology is called the product topology on this product of X_i's. Just some remarks. So, for us the box topology is not useful, and we will see reasons for this very soon. Throughout this course when we talk of a product of topological spaces, when i is an infinite set, we shall always be considering the topology $\mathcal{T}_2$. So, when we say the product topology on an infinite product of topological spaces X_i's, we shall always be referring to the second topology defined using the basis $\mathcal{B}_2$.

Let us make some more remarks. The following two remarks are obvious. One is when the index set is finite. If we take a set I of finite cardinality Then it is clear that the box topology is equal to the product topology. in fact $\mathcal{B}_1 = \mathcal{B}_2$ if the index set is finite.

So, that is a easy check when cardinality of I is infinite, we have $\mathcal{B}_2 \subseteq \mathcal{B}_1$ and we know $\mathcal{B}_1 \subseteq \mathcal{T}_1$. Recall that we have proved the lemma that if X is a topological space and $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X with basis $\mathcal{B}_1$ and $\mathcal{B}_2$, and if $\mathcal{B}_1 \subseteq \mathcal{B}_2$, then we get that $\mathcal{T}_1$ is contained in $\mathcal{T}_2$. So, using this lemma in our situation we have $\mathcal{B}_2 \subseteq \mathcal{T}_1$, this will imply that $\mathcal{T}_2$ is contained in $\mathcal{T}_1$. So, this is the product topology and this is the box topology. In fact $\mathcal{T}_1$ is much larger than $\mathcal{T}_2$.

As I had mentioned before we will almost never use the box topology in this course, we will always use the product topology. So, we have several examples. We can construct several examples of topological spaces. So, let us see some of the most important examples which we will encounter in this course. So, first is, we have the standard topology on the real line, we have the standard topology on $\mathbb{R}^2$ and $\mathbb{R}^n$.

So, on $\mathbb{R}^n$, let me make a remark on $\mathbb{R}^n$, we have the standard topology let us call this S and the product topology. Because, we can think of $\mathbb{R}^n$ as the product of $\mathbb{R}$, n times and each of these factor $\mathbb{R}$'s carries the standard topology. Claim: Let us call the product topology $\mathcal{T}$, we claim $S = \mathcal{T}$. So, the standard topology on $\mathbb{R}^n$ is equal to the product topology on $\mathbb{R}^n$ and an easy way to prove this is to show that: let $\mathcal{B}_!$ be a basis for S. We had already defined a basis, $\mathcal{B}_1$, for S using the sets, I do not remember the notation now, may be $S_\varepsilon(x\_)$, these epsilon squares around the point x, and let $\mathcal{B}_2$ be the basis for the product topology $\mathcal{T}$.

So, then show that $\mathcal{B}_1 = \mathcal{B}_2$. So, this will automatically imply that $S = \mathcal{T}$. So, in other words, this means that on $\mathbb{R}^n$, we have put two topologies, the first is a standard topology and the other is a product topology, and both these topologies agree, which is a nice thing. So, second we can take S^1, this is the unit circle in $\mathbb{R}^2$, points (x,y) in $\mathbb{R}^2$ such that $x^2+y^2=1$, with the subspace topology. So, this is the unit circle and similarly we can define S^n.

These are the unit spheres, those $(x_0, x_1, x_2, ..., x_n)$ in $\mathbb{R}^{(n+1)}$, such that summation i = 0, 1, .

.., n of $(x_i)^2$'s = 1, again with the subspace topology. So, here the subspace topologies are from $\mathbb{R}^2$ obviously and with the subspace topology from $\mathbb{R}^{(n+1)}$ respectively. The fourth example is the set of $n \times n$ matrices over $\mathbb{R}$. This is the set of $n \times n$ matrices with real coefficients. This set is in bijection with $\mathbb{R}^{(n^2)}$ and $\mathbb{R}^{(n^2)}$ carries the product topology.

So, using the topology on $\mathbb{R}^{(n^2)}$ and since $M_n(\mathbb{R})$ is in bijection with $\mathbb{R}^{(n^2)}$, we can transfer the topology from $\mathbb{R}^{(n^2)}$ to $M_n(\mathbb{R})$. In other words we can take a map $\phi:M_n(\mathbb{R}) \to \mathbb{R}^{(n^2)}$, and for every So, for every open set over here, we can take $\phi^{(-1)}(U)$. In other words, we define $\mathcal{T} = \{\phi^{(-1)}(U)$, where U is open in $\mathbb{R}^{(n^2)}$ with standard topology$\}$ Then $\mathcal{T}$ defines a topology on $M_n(\mathbb{R})$. In fact we can do this. So, if X is a topological space, we can do this generally, and $\phi:Y \to X$ is a bijection.

Then we can define a topology on Y using ϕ, let us say $\mathcal{T}_{(Y,\phi)}$ this topology will depend a priori on ϕ, this is defined as $\phi^{(-1)}(U)$'s, where U's are open in X. So, check that $\mathcal{T}_{(Y,\phi)}$ is a topology on Y. Let us look at the set $GL_n(\mathbb{R})$. This is the set of those matrices A in $M_n(\mathbb{R})$ such that $\det(A)$ is not equal to 0. We give this the subspace topology from $M_n(\mathbb{R})$.

So, $M_n(\mathbb{R})$, we have identified with $\mathbb{R}^{(n^2)}$ and using that we give a topology to $M_n(\mathbb{R})$ and $GL_n(\mathbb{R})$ is a subset of $M_n(\mathbb{R})$. So, we can take the subspace topology from $M_n(\mathbb{R})$ and put it on $GL_n(\mathbb{R})$. So, this makes $GL_n(\mathbb{R})$ a topological space. Similarly, we can take various subsets of $M_n(\mathbb{R})$ and put the subspace topology on all of these. We have the orthogonal groups, where $A^T*A = I$, then we have the special orthogonal groups where $\det(A) = 1$ also.

Next, we can put a topology on the complex numbers as follows. So, in the same way that we put the topology on $\mathbb{R}$. So, we take $\mathcal{T}$ to be the collection of sets $U \subseteq \mathbb{C}$ which satisfies the following condition: for all z in U, there exist $\varepsilon>0$ (which depends on z) such that (let me use a notation z for complex numbers) such that this open ball, of radius ε around z which is defined to be $\{y$ in $\mathbb{C}$ such that $|y-z|<\varepsilon\}$ So for every z in U, there should be an $\varepsilon>0$ such that this ball is contained in U. So, in other words, if these are the complex numbers, and this some U. We say that U is open in this topology if for any point z, there exist a small ball, a small disk around z which is completely contained inside U.

Alternatively we could have done the following: there is a bijection ϕ: $\mathbb{R}$ -> $\mathbb{C}$ which is $\phi(x,y) = x + iy$, and we can take the standard topology on $\mathbb{R}^2$ and since this is a bijection, we use it to define a topology on $\mathbb{C}$. So, $\mathcal{T} = \{\phi(U)$, where U is open in $\mathbb{R}^2$ in the standard topology$\}$ So, let me call this $\mathcal{T}_1$, and let me call this $\mathcal{T}_2$. It is an easy exercise to show that $\mathcal{T}_1 = \mathcal{T}_2$, both these topologies are going to be the same. So, we have put a topology on complex numbers and once again, we can, using the topology on $\mathbb{C}$, put a topology on $M_n(\mathbb{C})$, which is the set of $n \times n$ matrices with complex coefficients. So, how do we do this? It is in the same way as in $M_n(\mathbb{R})$.

So, we identify $M_n(\mathbb{C})$ in bijection with $\mathbb{C}^{(n^2)}$. Exactly as in the case of $\mathbb{R}$, what we can do is we can take a matrix a_{11} ... a_{1n}, a_{21} ... a_{2n} and so on. We can send this to first the row vector a_{11} ... a_{1n}, then we can write the next row a_{21} ... a_{2n}, this goes on and finally, we have the last row a_{n1} ... a_{nn}. So, we got done the same thing for real numbers also, for $M_n(\mathbb{R})$ also.

Let me say $M_n(\mathbb{R})$ to $\mathbb{R}^{(n^2)}$, we have the same bijection, and then we can pull back. So, using this bijection, $\mathbb{C}^{(n^2)}$ has the product topology, where each copy of $\mathbb{C}$ has a standard topology. So, by standard topology, I mean the topology defined over here as a standard topology, and then I am using this bijection to put a topology on $M_n(\mathbb{C})$. Just as we considered subsets of $M_n(\mathbb{R})$, we can consider the following subsets of $M_n(\mathbb{C})$, we can take $GL_n(\mathbb{C})$ instead, those A in $M_n(\mathbb{C})$ such that $\det(A)$ is not equal to 0 with the subspace topology from $M_n(\mathbb{C})$.

Then we can take $SL_n(\mathbb{C})$, the same. So, here $\det(A) = 1$, then we have the unitary groups A in $M_n(\mathbb{C})$ such that $A^*A = I$, and finally, we have the special unitary groups whose determinant is equal to 1 also. So, we have constructed various examples of topological spaces and for all the above examples we can study their topological properties. So, the main topological properties we will consider in this course are compactness, well first connectedness, path connectedness and compactness. These will be defined later on in this course. Now that we have constructed all these examples, that sort of brings the first part of this course to an end.

What we have basically seen is the definition of a topology and how to construct examples of topological spaces, and new topological spaces from ones which we already know. So, maybe we can phrase it as new topological spaces from old ones, and in order to study topological properties of these spaces, it is often good if we can relate them. This is just in the same way that when we study groups, in order to study group theoretic properties of groups, it can be very convenient if we have group homomorphisms, so we can study homomorphisms from one group into other groups and that helps us to say some things about the properties of our original group. So, in the same way, we will next introduce the notion of continuous maps between topological spaces. We will end this lecture here.