Point Set Topology
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Week 01
Lecture 05

So, let us return to the example of subspace topology and now we will consider the slightly more complicated example. So, let X be $\mathbb{R}$ with the standard topology. This is example 4, so this is lecture 5. And consider the inclusion map, so i: $\mathbb{R}$ -> $\mathbb{R}$^2 which sends x to (x,0) so we are just embedding the real line into $\mathbb{R}$^2 as the x axis. So, this is 0, and point x is going to (x,0). Let $\mathcal{S}$ denote the standard topology on $\mathbb{R}$, the real line and let $\mathcal{T}$ denote the standard topology on $\mathbb{R}$^2.

Let Y be equal to image of i, so Y simply the x-axis. and we can put, we can identify the real line with Y and we can put the subspace topology on Y. So, then so what this means is, so $\mathcal{T}$_Y is equal to U∩$\mathbb{R}$ where U is an open subset in $\mathcal{T}$. Here, we can take any open subset and we can intersect it with the x-axis, which means over here we will get the union of these intervals.

And this map is i, so this U∩$\mathbb{R}$, we can also view it as i^{-1}(U). So, we claim that we claim that $\mathcal{S}$ and this $\mathcal{T}$_Y are equal. So, these are the same topologies on the real line Let us prove this. So, instead of proving this directly we will prove two lemmas which will be helpful, and they will be helpful in other contexts as well. Lemma: Let X be a set and suppose there are two topologies on X, say $\mathcal{T}$_1 and $\mathcal{T}$_2.

So, let $\mathcal{B}$_i be a basis for $\mathcal{T}$_i, i =1,2. We are given two topologies on X, exactly as in this previous example, we have two topologies: One is $\mathcal{S}$, which is the standard topology, and the other is $\mathcal{T}$_Y which is coming as the subspace topology from the standard topology on $\mathbb{R}$^2. And we have $\mathcal{B}$_1 is a basis for $\mathcal{T}$_1 and $\mathcal{B}$_2 is a basis for $\mathcal{T}$_2. If $\mathcal{B}$_1 is contained in $\mathcal{T}$_2, then $\mathcal{T}$_1 is contained in $\mathcal{T}$_2. So, if the basis for $\mathcal{T}$_1 is contained in $\mathcal{T}$_2, then all of $\mathcal{T}$_1 is contained in $\mathcal{T}$_2.

That is the content of this lemma. Let us prove this. Let U be a subset of X which is open in $\mathcal{T}$_1, or which is simply an element of $\mathcal{T}$_1. We will show that U is open in $\mathcal{T}$_2. So, since $\mathcal{B}$_1 is a basis for $\mathcal{T}$_1, for every x in U, there exists a W_x which is in $\mathcal{B}$_1 such that this W_x is also a subset of U.

So, as a result, we may write U as the union of, so let I be the collection of such W_x for every x in U. So, for every x in U, we choose one such W_x, and we take I to be the collection of all these W_x's. So, then clearly U is equal to the union over the elements of

I of these W_x's. So, this inclusion is clear, as each W_x is contained in U and therefore the arbitrary union is contained in U, and this inclusion is clear, because given any x in U, it is in W_x. Now since each W_x is in $\mathcal{B}$_1, and $\mathcal{B}$_1 is a subset of $\mathcal{T}$_2, this implies that W_x is in $\mathcal{T}$_2, that is W_x is open.

Since $\mathcal{T}$_2 is a topology, and an arbitrary union of open sets is open, so this is by condition 3 in the definition of topology, the unions of open sets are simply open. So, this implies that this union over I of W_x's, is also in $\mathcal{T}$_2, which implies that U is in $\mathcal{T}$_2. Therefore, we have proved that given any U in $\mathcal{T}$_1, it is in $\mathcal{T}$_2. So, thus given any U in $\mathcal{T}$_1, U belongs to $\mathcal{T}$_2. This implies that $\mathcal{T}$_1 is contained in $\mathcal{T}$_2, which completes the proof of the lemma.

So, as a corollary, if with the same notation as in the lemma, if $\mathcal{B}$_1 is contained in $\mathcal{T}$_2 and $\mathcal{B}$_2 is contained in $\mathcal{T}$_1, then we get $\mathcal{T}$_1 = $\mathcal{T}$_2. So, we will use this corollary, and we need one more lemma, the proof of which is easy and left as an exercise. Let X be a set with a topology $\mathcal{T}$. Let Y ⊆ X be a subset and let $\mathcal{T}$_Y denote the subspace topology. So, if $\mathcal{B}$ is a basis for $\mathcal{T}$, then there is a obvious candidate for a basis for Y, $\mathcal{B}$_Y, which is defined to be U∩Y, where U is in $\mathcal{B}$, is a basis for $\mathcal{T}$_Y.

The proof is left as an exercise. Using these two lemmas, more precisely this corollary and this lemma we will show that Proposition: We will prove our claim that, let $\mathcal{S}$ and $\mathcal{T}$_Y be as described above then $\mathcal{S}$ is equal to $\mathcal{T}$_Y. So, recall what we are doing. $\mathcal{S}$ is a standard topology on the real line and $\mathcal{T}$_Y is the topology on the real line defined as follows. We can embed the real line into ℝ^2 as the x-axis, and we can take the subspace topology on the x-axis from the standard topology on ℝ^2 and we have to show that both these topologies are equal.

Proof: Recall that $\mathcal{T}$ is the standard topology on ℝ^2, and this had, as basis the collection S_ε(a,b) is equal to those points (x,y) in ℝ^2 such that absolute value of | a - x | < ε and | b - y | <ε. So, then $\mathcal{B}$_Y defined over here is equal to {Y∩S_ε(a,b)} So, on the other hand this is a basis for the subspace topology on the real line. We have the real line, This is the map "i" So, on the other hand, let $\mathcal{C}$ be the collection of B_ε(x) (here x belongs to ℝ, and ε > 0), union the empty set. So, here this B_ε(x) is contained in ℝ and recall that B_ε(x) is the interval (x-ε,x+ε), this is the subset of the real line. So, then $\mathcal{C}$ is basis for the standard topology on ℝ, which is $\mathcal{S}$.

We want to show that $\mathcal{S}$ = $\mathcal{T}$_Y. By the corollary, it is enough to show that $\mathcal{B}$_Y = $\mathcal{C}$. So, $\mathcal{C}$ is a basis for $\mathcal{S}$ and $\mathcal{B}$_Y is a basis for $\mathcal{T}$_Y. We want to show that $\mathcal{S}$ = $\mathcal{T}$_Y and therefore we will apply this corollary. So, let us check that $\mathcal{B}$_Y is indeed equal to $\mathcal{C}$.

Let us look at what the elements of $\mathcal{B}_Y$ look like. So, given if (a,b) is here then it may happen that $S_\varepsilon(a,b)\cap Y$, Y is the x-axis, it may happen that $S_\varepsilon(a,b)\cap Y$ is the empty set. Here, that is one possibility. The other possibility is (a,b) is here, and then $S_\varepsilon(a,b)\cap Y$ (intersects the x-axis) is going to be $B_\varepsilon(a)$. So, that is the only two possibilities here.

Note that $\mathcal{B}_Y$, or rather note that for every $S_\varepsilon(a,b)$ in $\mathcal{B}_Y$, either $S_\varepsilon(a,b)\cap Y$ is the empty set or $S_\varepsilon(a,b)\cap Y$ is exactly $B_\varepsilon(a)$. So, we are identifying Y, the x-axis with the real line using this embedding i and because of that, this $S_\varepsilon(a,b)\cap Y$ is going to be identified with this $B_\varepsilon(a)$ over here. Therefore so thus it is clear that this $\mathcal{B}_Y$ is contained in this collection $\mathcal{C}$, because $\mathcal{C}$ contains $B_\varepsilon(x)$ for all x in $\mathbb{R}$, and conversely, if we take some element $B_\varepsilon(x)$ in $\mathcal{C}$, then we can take, we have x, we can take the ε-square around $(x,0)$. So, what I am saying is $S_\varepsilon(x,0)\cap Y$ is equal to $B_\varepsilon(x)$. So, thus $\mathcal{C}$ is also, and of course the empty set is contained in $B_\varepsilon(x)$ because we can just take some $S_\varepsilon(x,y)$ over here, and when we intersect this with Y, we get the empty set.

So, the only elements of $\mathcal{C}$ is either the empty set which is contained in $\mathcal{B}_Y$ or it is of the type $B_\varepsilon(x)$, which we have just shown is contained in $\mathcal{B}_Y$. So, therefore we have proved both inclusions which implies that both these bases are equal, which implies that the subspace topology is equal to the standard topology on $\mathbb{R}$. In the same way, we may embed $\mathbb{R}^2$ into $\mathbb{R}^n$ as a hyperplane. So, I can take (x_1,x_2) and this maps to $(x_1,x_2, 0, .$

$.., 0)$ and in And in the same way, one may show that, the standard topology in $\mathbb{R}^2$ is the same as the subspace topology. so using i, we can identify $\mathbb{R}^2$ with its image, and using that identification, we can take the subspace topology on $i(\mathbb{R}^2)$ and transfer it to $\mathbb{R}^2$. So, and then $\mathbb{R}^2$ now has two topologies and both these topologies are the same. So, we have seen the subspace topology and next we want to define product topology. So, this ends our discussion on subspace topology.

So, the next thing we want to define is a topology on the product of two sets, each of which have a topology. But before that we need, so next let me just write down, Suppose X_1 and X_2 are two topological spaces. Then, we next want to define a topology on $X_1 \times X_2$. So, there is a natural way to do this, and let us see how to do it. So, let us first prove a lemma: Let X and Y be two topological spaces, and let $\mathcal{B} \subseteq P(X\times Y)$ be the collection of subsets defined as: $B = \{U\times V\}$, where U is in the topology on X (is open in the topology on X), and V is open in the topology on Y.

So, when we say X and Y are topological spaces it means that there have topologies, which we denote $\mathcal{T}_X$ and $\mathcal{T}_Y$. So, this is the definition of $\mathcal{B}$, and what we want to show is, the content of the lemma is, then $\mathcal{B}$ satisfies the two conditions in the proposition on generating topologies. So, I am referring to this proposition here, oops that was in the previous lecture, sorry. So, in the previous lecture we proved this proposition.

So, let me just recall it. So, recall the two conditions. Proof: Recall the two conditions we needed to check where the following: First, when we take the union of all W in $\mathcal{B}$, then we get the entire set. So, in this case, that is $X \times Y$. The second condition is suppose W_1 and W_2 are in $\mathcal{B}$, and x is an element in their intersection. Then there is a W in $\mathcal{B}$ such that x is in W and W is contained in $W_1 \cap W_2$.

So, if these two conditions were satisfied, then we saw that this collection $\mathcal{B}$ defined a topology, which we denoted $\mathcal{T}_{\mathcal{B}}$. Let us just check that these two conditions are going to be satisfied. So, first note that since X belongs to $\mathcal{T}_X$ and Y belongs to $\mathcal{T}_Y$, this implies from the definition of $\mathcal{B}$, $X \times Y$ belongs to $\mathcal{B}$. So, therefore clearly this union of W's contains $X \times Y$, and of course it is a subset of $X \times Y$. Therefore clearly this union of W's is equal to $X \times Y$.

This proves (a). (a) is indeed true. Let us check the second condition. So, for (b), we are given W_1 and W_2 in $\mathcal{B}$. This implies from the definition of $\mathcal{B}$, recall that $\mathcal{B}$ is defined as above. So, W_1 is equal to $U_1 \times V_1$, and W_2 is equal to $U_2 \times V_2$, where U_i's are from $\mathcal{T}_X$ and V_i's are in $\mathcal{T}_Y$.

So, let us take a point in $W_1 \cap W_2$ which looks like (a,b) which is there in $(U_1 \times V_1) \cap (U_2 \times V_2)$ Now this implies that a belongs to U_1 and U_2 and b belongs to V_1 and V_2, which in turn implies that a belongs to $U_1 \cap U_2$ and b belongs to $V_1 \cap V_2$. Now since $\mathcal{T}_X$ and $\mathcal{T}_Y$ are topologies note that $U_1 \cap U_2$ is in $\mathcal{T}_X$ and $V_1 \cap V_2$ is in $\mathcal{T}_Y$. So, thus we have (a,b), this is in $(U_1 \times V_1) \cap (U_2 \times V_2)$ which is an element of $\mathcal{B}$, because this is in $\mathcal{T}_X$ and this is in $\mathcal{T}_Y$ and this is in turn contained in $U_i \times V_i$ for $i = 1,2$. In other words (a,b) is contained in $(U_1 \cap U_2) \times (V_1 \cap V_2)$ which is contained in $(U_1 \times V_1) \cap (U_2 \times V_2)$ but this is W_1 and this is W_2. So, therefore we can take this set as W.

So, given any (a, b) in the intersection, we have found W in $\mathcal{B}$ such that (a, b) is in W and W is contained in the intersection. So, this proves that $\mathcal{B}$ satisfies the two conditions to generate a topology, and so it generates a topology, which we to generate a topology and so it generates a topology which we denote $\mathcal{T}_{\mathcal{B}}$ which has $\mathcal{B}$ as a basis. So, this ends the proof of this lemma. Let us just define the product topology: let X and Y be topological spaces. Then the topology defined on $X \times Y$ in the previous lemma is called the product topology on $X \times Y$.

So, a basis for this topology is given by sets $U \times V$, where U belongs to $\mathcal{T}_X$ and V belongs to $\mathcal{T}_Y$. So, we will end this lecture here.