Point Set Topology
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Lecture 35

In this lecture we will show, in this lecture and the next one that there is a large class of technological spaces which admit many continuous maps from X to Y or many which admit many continuous functions. So, for that we can recall that a Hausdorff space is one in which we can separate points by open sets. So, what is this mean? We call a space X to be Hausdorff if, given any two points on X, we can find two open subsets U and V. So, U contains x and V contains y and U intersection V is empty. So, similar to this we have the following definition. A and B contained in X.

So, here is our X, we take two let us say disjoint subsets A and B, this is A this is B and these are closed. Then there should be open subsets U and V such that that A is contained in U, B is contained in V and U intersection V is empty. If this happens for any two pairs of disjoint closed subsets then we say that A is normal. So, let us see this proposition which says that there are lots of normal spaces.

So, every metric space is normal. Let us see a proof of this. Let X be a metric space and let A and B be closed subsets of X. Recall the functions d_A from X to ℝ and d_B from X to ℝ. In fact we had for any subset Z of X we had defined this map d_Z from X to ℝ as follows: The distance of x from Z is defined to be the infimum over all z in Z out this distance of x from z.

So, if our X is this like this, and x is here, and let us say this is our Z. We want to compute the smallest distance. We take points z in Z, compute the distance of x from z and we take the infimum in that collection. That is defined to be the distance of x from Z. So, we had seen that this function is continuous for any subset.

Let us prove this easy proposition, or rather lemma: if Z is closed, if Z is closed then d_Z(x)=0 iff x belongs to Z. Proof: so first assume that this distance is 0. This implies that the infimum of z in Z is equal to 0 this implies that there is a sequence z_n such that the distance of x from z_n is converging to 0, this implies that z_n converges to x. This implies that x belongs to Z closure, but as Z is closed, it is equal to Z. So, this implies x belongs to Z, and obviously if x belongs to Z, then the distance of x from Z is 0, because x appears in this collection.

So, d_Z(x)=0. Having proved this lemma, we will use this lemma. Now let us return to a proposition. We are given two disjoint closed subsets A and B, disjoint closed subsets, and we have to find open subsets disjoint, one which contains A and the other which contains B. So, let us say this is our A and let us say this is our B.

So, for each a in A let ε_a be defined to be the distance of a from B divided by 4. So, we can take any a over here. Let us take a over here, and let us compute the distance of a from B right and we take this ε_a to be one-fourth of that distance. So, similarly for b in B let ε_b be the distance from A of b divided by 4. So, define open sets.

Let U be equal to the union over a in A of ε-balls around a of radius ε_a. And so, we are taking this ball of radius ε_a and taking the union over all these a's, and similarly V to be union of b in B the balls of radius ε_b. note that ε_a is positive or else, as B is closed, by the lemma we will have a is in B, which is contradiction as A and B are disjoint. Similarly, ε_b is also positive. Thus, we get open sets U and V such that U contains A and V contains B.

So, we claim that U intersection V is empty. If not, our U is going to be something like this. So, V is going to be something like this, and U is going to be a union of something like this. If not, then there exists y in U intersection V. There exists a in A and b in B such that y is in $B(a,\varepsilon_a)$ intersected $B(b,\varepsilon_b)$.

Now note that the distance of b from A is less than equal to the distance of this, which by triangle inequality is less than equal to $d_A(y)+d_B(y)$. As y is over here, this quantity is less than ε_a. So, this is strictly less than $\varepsilon_a+\varepsilon_b$. So, this is strictly less than ε_a as y is here and this is strictly less than ε_b as y is here. So, I should make this ε_a.

We may assume that without any loss of generality that ε_a is less than equal to ε_b, and then this is going to be strictly less than ε_b. This will imply that the distance of b from A is less than equal to ε_b, but now remember what was ε_b. ε_b was equal to the distance of b from A by 4. So, this implies that $d_A(b)$ is equal to 0, and as A is closed, this implies that b belongs to A which is a contradiction. So, thus U intersection V is empty.

This completes the proof of the proposition. So, this shows that there are lots of normal spaces, in fact any metric space is normal and let us prove another lemma. So, we will prove a few results which are in preparation towards proving Urysohn's lemma. So, let X be a normal space, and let A be a closed subset. We have a closed subset A, and we have an open subset W.

So, then this lemma says that there is a V, an open subset, such that A is contained in W. Then there is an open subset V such that A is contained in V, is contained in V closure, is contained in W. Let us prove this lemma. The sets A and X\W, as, W is open, so, X\W is closed. So, X\W is this region outside W, including this boundary, are disjoint closed subsets.

Thus there are open subsets U and V such that U contains X\W and V contains A, this is using normality. So, every time we have disjoint closed subsets, we can separate these using disjoint open subsets. So, that is the definition of normality. So, now, we claim that V closure intersected X\W is empty So, why is that? Because if not, let us just make a picture.

So, U is this region. This region is U, if not then there exists t in X\W such that t belongs to V closure, but this is not possible as U is an open set containing t such that U intersection V is empty. What does it mean for t to be in the closure of V? It means that every open subset containing t has to meet V. That is not possible. So, this implies that V closure is does not meet X\W, which implies that V closure is completely contained, thus A is contained in V is contained in V closure. So, this proves the lemma and the final proposition Let X be a metric space.

Let A and B be disjoint closed subsets. Then there exists a continuous function f from X to [0,1] such that $f(A)=0$ and $f(B)=1$. So, we have our X, we have two disjoint closed subsets A and B. We are claiming that there is a map to [0,1] which takes A to 0 and takes B to 1.

Let us prove this. So, for metric spaces this is very easy to prove. So, take f(x) to be equal to $d_A(x)/\{d_A(x)+d_B(x)\}$. So, now, let us look at the function. The functions d_A and d_B are continuous. This implies that d_A+d_B is continuous and since A intersection B is empty, d_A+d_B for any x in X, when we look at $d_A(x)+d_B(x)$, this has to be positive because if it is 0, then it will mean that we will have $d_A(x)=d_B(x)=0$ which will imply that x belongs to A intersection B as both are closed.

So, thus this function d_A+d_B takes positive values. This implies that this x goes to $d_A(x)+d_B(x)$ is continuous, and since the product of continuous functions is continuous, this implies that f(x) is continuous. Now, if x belongs to A then clearly f(x) is equal to 0 and if x belongs to B then $f(x)=d_A(x)/\{d_A(x)+d_B(x)\}$ but this is 0 because x belongs to B. So, this is equal to 1. So, this completes the proof of the proposition.

So, in the next lecture we will prove Urysohn's lemma. So, Urysohn's lemma proves the same proposition, but it removes the hypothesis that X is a metric space and it replaces metric space by being normal. So, we will end this lecture here.