Point Set Topology
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Lecture 34, Part III

Now we will see an application of the above result. So, let G be equal to GL(n,ℝ). Then G is a group and has a topology. We claim that G is a topological group. Let us check that. We need to show that m from GL(n,ℝ)xGL(n,ℝ) to GL(n,ℝ), the multiplication map is continuous.

Now note that this has a product topology. In particular, we know that the projection maps are continuous. So, we can look at the projection maps GL(n,ℝ) this is some p_i and let us say on GL(n,ℝ), we have these continuous maps which project on to the coordinates. Let us call this q_{ij}'s.

So, given a matrix A, this map q_{ij} it sends it to a_{ij}. So, this implies that the composite of these two is continuous. Therefore, we get that from GL(n,ℝ)xGL(n,ℝ) to ℝ. So, (A,B) maps to a_{ij} or similarly b_{ij}, these are continuous. And this multiplication map m, the coordinates of m are polynomials in the functions a_{ij} and b_{ij}.

And we have seen that if we take a collection of continuous maps and we form any polynomial using those continuous maps, then the resulting map function is continuous. This implies that m is continuous. So, similarly let us take GL(n,ℝ), let us look at the inverse map this A goes to A^{-1}. So, we know the inverse is given by this formula 1/det(A) into adjoint of A transpose. So, now, once again the determinant does not vanish on GL(n,ℝ).

So, 1/det(A) is a continuous function and this adjoint of A transpose, each of the coordinates is a continuous function in the coordinates of this GL(n,ℝ) and therefore, the product is a continuous function. This implies that GL(n,ℝ) is a topological group. Now, let P be this parabolic subgroup. Here we have a rxr matrix here, we have 0 and these can be anything right. We are looking at all P contained in G, G=GL(n,ℝ), be this subgroup.

All elements in G which look like this. So, clearly P is a closed subgroup. It is given as the inverse image of the projection onto this piece, projection of 0 on the inverse image of 0, under the projection onto this piece. Therefore, P is a closed subgroup. So, this implies that.

The previous discussion shows that G/P is a Hausdorff topological space. Now we want to identify G/P, we want to give a nice description of the points of G/P which is what we are going to do next. So, for that consider the map ϕ from G to the r dimensional subspaces of $\mathbb{R}^n$. So, let us call this set script G. Here what is the map? So, let us take a matrix A and let us write A as $[v_1, v_2, ..., v_n]$ v_i's are the columns of A. Since A is in $GL(n,\mathbb{R})$, this implies that the column vectors of A are linearly independent. So, we just send it to the span of v_1 upto v_r. Now, we claim that $\phi(A)$ is equal to $\phi(B)$ iff there exists T in P such that, our parabolic subgroup, such that B=AT. So, let us see this.

So, we take two matrices A and B. So, B is equal to AT. So, then clearly $\phi(A)=\phi(B)$ because if we take B equal to, write it as $[w_1,w_2, ..., w_n]$. So, this is equal to v_1, v_2 upto v_n times this parabolic subgroup, this rxr matrix, zero, we can have anything right. So, this will show that this is AT. So, this will imply that the w_1, upto w_r are linear combinations of v_1 upto v_r, and both are r dimensional subspaces. This implies that the span w_1 upto w_r is equal to span v_1 upto v_n. So, this shows that $\phi(A)$ is equal to $\phi(B)$ next let us assume that $\phi(A)$ is equal to $\phi(B)$ and then show that, if $\phi(A)$ is equal to $\phi(B)$, then we have the span of w_1 upto w_r is equal to the span of v_1 up to v_r.

Now, since the w_1 upto w_n they span $\mathbb{R}^n$, and v_1 upto v_n also span $\mathbb{R}^n$. As w_i's and v_i's are a basis for $\mathbb{R}^n$, this implies that there exist a unique matrix T in $GL(n,\mathbb{R})$ which can be obtained as follows: So, what we do is for each w_i, we can write it as a unique linear combination of the vectors v_1 upto v_n So, if I write w_1 as $\lambda_1v_1+\lambda_2v_2+...+\lambda_nv_n$. Then the first column of T will be λ_1 upto λ_n, and this matrix T clearly has been $GL(n,\mathbb{R})$ because we can do the same. We can replace the roles of W and V in the composition, the resulting matrix T' that we get that is going to compose with T to be identity. So, the i^{th} column of T is the unique way to write w_i as a linear combination of v_j's. We just have to show that T is in P. So, note that since we are given this information, that implies that w_i is in the span of v_1 upto v_r.

So, this implies that the i th column of T is going to look like, this is r right and then everything will be 0. And this happens for i lying between 1 and r. So, this clearly implies that T belongs to P. Clearly, this proves this claim, we have proved this claim. Next, the second point we want to show is, clearly ϕ is surjective.

So, as ϕ is this map from $GL(n,\mathbb{R})$, a matrix A goes to the span of the first r column vectors.

Now, given any r-dimensional subspace of $\mathbb{R}^n$, we can choose a basis for that r-dimensional subspace and v_1 upto v_r and we can extend it to a basis of $\mathbb{R}^n$, and correspondingly we can get a matrix A. As any basis v_1 upto v_r for an r-dimensional subspace can be extended to a basis v_1 upto v_n for $\mathbb{R}^n$. So, which implies we can take the matrix A to be $[v_1,v_2,...,v_n]$. So, this shows that ϕ is surjective. So, this implies that when we put together 1 and 2, one can easily check that we have this map ϕ to script G right and this factors through G/P, and this map is a bijection let us call this ϕ_0. This easily follows from 1 and 2, that ϕ_0 is a bijection. So, thus the points of G/P are in bijection with r-dimensional subspaces of $\mathbb{R}^n$. So, which means using this bijection we can give this space of r-dimension subspaces of $\mathbb{R}^n$, this set G, we can give G a topology.

So, we identify G with G/P using this ϕ_0 and a subset of G is open if and only if its inverse image under ϕ_0 is open. Clearly so G/P is a topology and using this bijection we can transfer the topology. This topological space is called the Grassmannian of r-planes in $\mathbb{R}^n$ and its denoted Grassmann(r,n). So, by the proposition we proved, we know that G/P is Hausdorff, we get that Grassmann(r,n) is Hausdorff and further note that also $GL(n,\mathbb{R})^+$ surjects onto this Grassmann(r,n) because why is that? So, given an r-dimensional subspace we chose a basis v_1 through v_r for the subspace and we extended to a basis.

So, we get this matrix A. If the determinant of A is negative, then we just take the last column and put a minus sign, take instead of v_n we just take $-v_n$. And we had seen as $GL(n,\mathbb{R})^+$ is path connected, implies that this Grassmann(r,n) is path connected. And similarly, the Gram-Schmidt orthogonalization process implies that O_n, the orthogonal matrix surjects onto this Grassmann(r,n). So, why is this? So, we have O_n this sitting inside $GL(n,\mathbb{R})$, and here we have this map ϕ_2. So, we claim that this composition is surjective.

Why is that? If we take any r-dimensional subspace, first, we can choose a basis v_1 up to v_r, then applying the Gram-Schmidt orthogonalization process, we can get an orthonormal basis for this. And then we can extend this orthonormal basis for the subspace to an orthonormal basis of $\mathbb{R}^n$. So, let me just write that, and extend this to an orthonormal basis for $\mathbb{R}^n$. So, we get w_1 upto w_n. Then the matrix A is this $[w_1, w_2$ upto $w_n]$.

Since, the w_i's are orthonormal, we will get that A^T*A is equal to identity, which will imply that A is in O_n. And now, this G is actually equal to G/P. Now as O_n is compact, since it is a closed and bounded subspace of $\mathbb{R}^{n^2}$, this implies that, and O_n surjects on to G/P, this implies that G/P, being the image of a compact space, is compact. Thus we have proved that the Grassmannians is Hausdorff, path connected and compact. So, we can also construct do a similar construction for complex Grassmannians.

So, there we will take r-dimensional complex subspaces of $\mathbb{C}^n$, and in the same way we can show that this is Hausdorff, path connected and compact. And before I end, let me just mention that when r=1, the spaces Grassmannian(1,n) is also known as the projective space and denoted $P(n,\mathbb{R})$ or $P(n,\mathbb{C})$. So, if we take complex projective spaces, depending on the underlying field, we can do this entire construction for the real numbers or the complex numbers. So, this is the real projective space and this is the complex projective space. We will end this lecture here.