Point Set Topology
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Lecture 33

Let us begin this lecture. So, in the last few lectures, we discussed the one point compactification of a noncompact and locally compact topological space. So, let me make a remark on what happens when X is compact. So, if X is compact, so in the results, we are assuming that X is not compact, but X is locally compact. But what happens when X is compact? If X is compact, then X is also locally compact. This is because of silly reasons, as for each x in X, we can take the open set U to be X.

Remember the definition of a locally compact topological space, we need that every point x in X, our space should have a neighborhood U, such that the closure of U is compact. If we take U to be X in this case, since U closure is going to be X, and X is compact, so this will do the job. A compact topological space is obviously locally compact and if you consider the one point compactification of X then we construct the one point compactification in the same way. Our X hat is going to be X disjoint union {p_0} and the topology is given in the same way that we did in the noncompact case.

And we can easily check that, it is easily checked that this point p_0 will be an open and closed subset. So, if our X is for instance the sphere, which we know is compact because it is closed and bounded inside ℝ^3, then X hat will simply be the sphere disjoint union a point away from this. The conclusion is that we have this inclusion of X into X hat, then i(X) is not dense in x hat. So, in any case we wanted to compactify things, So that we get a compact space, and so if our space is already compact, so then it is not very interesting, so this is just a remark. So, in the previous lecture we proved the following, so let me just make this p_0 over here, p_0 is going to be this point.

So, we had proved the following two results: (A): the first result we proved is let X be noncompact and locally compact topological space. Then there exists a Hausdorff and compact topological space X hat, and an inclusion i from X to X hat, such that the following four things happened: X hat is simply i(X) disjoint union {p_0}, i is continuous, i(X) is open in X hat and i(X) is dense in X hat. So, if X is already compact, then everything else will happen, only the fourth condition will fail. And another result we have proved is the uniqueness of the one point compactification, Let T, so let me just correct this, I had, in the previous lecture, where we proved this result (B), I had said that X has to be compact, but that is not necessary. Let T be a Hausdorff space, I just want to correct this, so in the previous lecture I had stated this to be compact, so compactness is not required.

We only need T to be a Hausdorff space. So, assume that there is a continuous map j from X to T such that j(X) is an open subset of T, and this map j from X to j(X) is bijective and is a homeomorphism. So, then we considered the following map f from T to the one point compactification of X, so X is a noncompact and locally compact topological space. So, X we have j over here and we have X hat, this is our i and we are going to define f as follows, so the picture we had made was the following. We took this open disk, and let us say we put it into this closed disk, this j and let us say i take an equator over here, so this is the one point compactification.

This is the additional point p_0, this i, so this equator is going to look like something like this. All the points at infinity, all these boundary points, like everything towards the boundary is going to get collapsed to p_0. As we move further and further away, as we move towards the boundary of, so all those points are going to get collapsed to p_0. So, we define the map f as follows: if T is in j(X), then define f(T) to be equal to ioJ^{-1}(T). This makes sense because j is a bijection from X to j(X).

If T is in j(X), then there is a unique j^{-1}(T) and I can just apply i to it and if T does not belong to j(X), then define f(T) to be equal to p_0 and the assertion of the proposition was then f is continuous. Let us see some applications of this proposition, so this proposition we have proved in the previous lecture. So, as an application, let us prove that the one point compactification is unique. So, what do you mean by that? Suppose there are two compactifications, so we have X over here and here we have this inclusion i from X to X hat. Now, suppose there is a j from X to T such that we have the following properties T is compact, j(X) contained in T, is open and T\j(X) is just one point.

So, let us call that point p_oo over here or let us call this one point q_0 and j from X to j(X) is a homeomorphism. In this case T is another compact space which is obtained by adding q_0, which is another compact topological space and T\j(X) is just this one point and j(X) is embedded inside T as an open subset. So, then by our previous proposition there is a map f, so we may write T as j(X) disjoint union {q_0} and we may write X hat as i(X) disjoint union this {p_0}. So, by our construction, definition of the map f is ioj^{-1}. It is a bijection and here it takes this one point outside j(X) and that is sent to p_0, that is by definition.

This implies that, clearly f is a bijective map and by the previous result, by the previous proposition f is continuous. Moreover using the result as T is compact and X hat is Hausdorff we get that f is a homeomorphism. So, here we have used the following result, f from X to Y is a bijective continuous map, where X is compact and Y is Hausdorff. Then f is a homeomorphism, let me make a remark, it is important that Y should be Hausdorff, important that Y is Hausdorff, which is an assumption throughout in the rest of this course. Unless it is stated explicitly that a space may not be Hausdorff because otherwise we can take the identity map from X to X and give this X the trivial topology.

For instance we can take X to be S^n, so here we can give X the usual topology and here we can give X the trivial topology. The identity map is obviously going to be continuous because here the target has the trivial topology, but obviously this map is not Hausdorff, even obviously this map is not a homeomorphism, even though X is compact. Before we proceed, let us remark that there is an important class of spaces which is not locally compact. Many of the spaces we have encountered in this course they are all subsets of $\mathbb{R}^n$ and $\mathbb{R}^n$ is locally compact, and many of the spaces we have seen in this course are locally compact, but there is a big collection of spaces which is not locally compact. Namely the infinite dimensional Hilbert spaces, so let us see why this is as an application of what we have learnt, let us see why this is not locally compact.

So, if X is locally compact, so if you do not know what Hilbert spaces are you can just forget about this example. So, then there is an r, positive, such that the open ball, the closure of this ball around of radius r, around the origin there is those x in X such that $||x||<=r$ is compact, but this is not possible. Now this is a compact metric, if this is compact, then this will be a compact metric space and therefore every sequence will have a convergence of sequence, but this is not possible as the sequence, we can take these vectors e_i's, the e_i's form a orthonormal basis for this Hilbert space $\{e_i\}$ does not have a convergent subsequence. So, this contradicts the result we proved that in every compact metric space, every sequence has a convergent subsequence. This brings us to an end of our discussion on locally compact topological spaces.

Next we want to describe what the quotient topology is. So, let us begin with an example from group theory which you would be familiar with. Let G be a group, or rather a concept from group theory, and let N contained in G be a normal subgroup. Then we have the first we have the set G mod N which is the set of equivalence classes G/N, where we define $x \sim y$, x and y are in G, if and only if $y^{-1}x$ is in this normal subgroup. So, instead of writing G mod equivalence, we often write G/N.

Moreover, we can give G/N or G mod equivalence, whichever way you want to write it, a group structure such that the natural map, let us call this π, this is just a map of sets a-priori to G mod equivalence, which takes an element x to its equivalence class. In this case the equivalence class is the coset of N. This natural map, so we can give G/N a group structure such that this natural map becomes a group homomorphism, that is one thing further. So, that is one nice thing which happens when N is a normal subgroup, but there is another nice thing which happens. This map π has the following property.

If we have a group homomorphism which is constant on equivalence classes. Then we get an induced map of sets so we have G to H, we have f and f is constant on equivalence classes. So, which means when we look at G mod equivalence, we get a map of sets, let us call it f bar and the important thing is that f bar from G mod N to H is a group homomorphism. Note that since f is a group homomorphism, f is constant on equivalence classes iff the normal subgroup N is contained in the kernel of f. So, in the next lecture, we will see an analog of this concept for topological spaces.

Before we end this lecture let us see one very standard application of this corollary that we saw of the uniqueness of one point compactification. Let X be equal to $\mathbb{R}$ So, then X is not compact and X is locally compact. Now we know that, recall the stereographic projection. So, what did we do? We had $\mathbb{R}^n$ and we took the sphere, and we deleted the north pole for instance. And for any point x on the sphere we joined p with x and we get a unique point y.

So, the stereographic projection ϕ is from $S^{n}\setminus\{p\}$ to $\mathbb{R}^n$. And we had seen that ϕ is a homomorphism, in particular if we let t to be S^n and $j=\phi^{-1}$, the inverse of this homomorphism. So, j is ϕ^{-1} from $\mathbb{R}^n$ to S^n. So, then the hypothesis of the previous corollary are satisfied. Because we are taking T to be $S^n\setminus im(\phi)$, is just the one point, the north pole, and this will imply that, this will force that the map from the induced map f that we get from S^n to X hat is a homomorphism Thus the one point compactification of $\mathbb{R}^n$ is S^n.

So, we will end this lecture here.