# Point Set Topology
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**Lecture 32**

In the previous lecture we saw that, given a topological space, a locally compact topological space X, if it is not compact then we can compactify it. What does it mean by saying that we can compactify it? It meant that we can construct a compact topological space X hat such that X is sitting inside X hat as a dense subspace. Recall that when we say a compact topological space X, in our definition of compactness we say compact topological space X hat, in our definition of compactness we need X hat to be Hausdorff, and we had proved that X hat is Hausdorff as a dense subset, as a dense open subset. In fact, X hat we obtained is just X disjoint union {p_0}. So there is a picture we can keep in mind, so let us take the real line, the real line is locally compact, but it is not compact, and we can take this point p_0 over here. So when we compactify the real line let us see what we get.

So instead of making the picture like this, we will make the real line like this and this point p_0 is here. Recall that there were two kinds of open subsets, one kind was type (A) open subsets were open subsets of X, and the type (B) open subsets were as follows: We take any compact subset of ℝ, so for instance we can take this union of two compact subsets of the real line, so that is going to be compact, and we just take the complement of compact subspaces, complements in X hat of compact subspaces of X. Or in other words what we have specified is what are the open subsets, the open subsets which contain p_0 are exactly complements in X hat of compact subspaces of X. So in the same way we can see another example of ℝ^2.

So instead of making ℝ^2 like a plane, we can make it as a sphere. From this in the picture it is clear that one point compactification seems that of ℝ should be S^1, and similarly if we take ℝ^2, I should make a dotted line. So this is our ℝ^2 and we can imagine the point p_0 over here so ℝ^2 is also locally compact and therefore we add this point p_0. So what are the neighbourhoods/open subsets which contain p_0? These are the open subsets of type B, so these will be, we can take any compact subset of ℝ^2, let us say this is D, so then D will be somewhere over here, and we just take the complement of D, so this (X hat)\D. So it is useful to keep this picture in mind.

Let us begin today's lecture, so let us write down what we proved last time. Let X be a locally compact topological space which is not compact. Then there is a compact topological space X hat, along with an inclusion i from X to X hat, such that X hat is just i(X) union a point, which we call p_0. 'i' is continuous, i(X) is an open subset of X hat and i(X) is dense in X hat. Let us make the following easy remark: Note that i from X to i(X) is a homeomorphism, So we have already seen that i is continuous, and the image of i lands inside i(X) and it is a bijection, so you have already seen or we know that i is a bijective continuous map.

So thus to show that i is a homeomorphism, it is enough to show that if U contained in X is open then i(U) is open in i(X), but now note that i(U) is the same set U, and as U is a subset of type (A), is an open subset of X hat of type (A), this implies that i(U) is open in X hat and as i(U) is contained in i(X), and so also in i(X). Thus i is a homeomorphism. This theorem, we had proved last time, and in the previous lecture I had mentioned that this particular one point: there could be many compactifications of a locally compact topological space but this particular compactification has a very nice and special property, which is what we will see in this lecture. So, the topological space X hat has the following property. Proposition: Let T be a compact topological space, which contains X as an open subset.

So what is the meaning of this sentence? This means that there is an inclusion j from X to T, it is an inclusion of sets such that j(X) containing T is an open subset and j from X to j(X) is a homeomorphism. There is an inclusion such that, let me also write j is continuous. Then there is a unique map, so what we have is we have X over here and we have this inclusion j, is a continuous map from X to T and the image, so let us say the image is j(X), and this is a subset of T. There is an open subset and we have this map, like this, and we also have the map i also satisfies these properties There is X hat that is also open in X hat. So then there is a unique map f from T to X hat, such that the following two things happen: foj is equal to i, and f(T\j(X)) is equal to this p_0.

So X hat has this point at infinity which is p_0. So let us try to see what this means by means of an example. So let us say X is this open disk, and let us say i embeds the open disk into this closed disk. On the other hand the one point compactification is j, this is the one point compactification looks like, so I have to add a point at infinity, so it will look like something like this. So this point is p_0.

So what is f doing? f is collapsing all of the boundary. So f collapses the set T\j(X), which in this case, is this black boundary, to the point p_0 and f is completely determined at other points by the condition foj = i. So this in this example f does this. Let us try to prove this proposition, which is that there is a unique such map f. Of course f is continuous there is a unique continuous map that is the assertion.

So the map f has been defined theoretically on all of T. So this map has been defined set theoretically on all of T. So how do we do it? On j(X), define f to be, j is a homeomorphism from X to j(X), so we apply j^{-1}(X), and then we apply, this goes to i(X). This makes sense as j from X to j(X) is a homeomorphism. So we do not need homeomorphism right now, so we just need bijective, is a bijection.

We just need bijection right now, because as of now we are really trying to say that f is defined set theoretically, and outside on T\j(X), we have defined f to be the constant map to p_0. Thus the map of sets f is completely determined and so it is unique. So we only have to show that f is continuous. So for this, let V contained in X hat be an open subset So let us first assume that p_0 does not belong to V. In that case, the open subset is contained somewhere over here.

So then it is easily checked that f^{-1}(V) is completely contained inside j(X). So as a result, this shows that f ^{-1}(V) is actually equal to f^{-1}(V) restricted to j(X). Now note that f^{-1} restricted to j(X) is from j(X) to X under j^{-1}, and this composed with i to i(X), and as X to j(X) is a homeomorphism, this implies that j^{-1} is continuous. This implies that ioj^{-1} is continuous. This implies that f restricted to j(X) is continuous.

So this implies that, combining these we get that f^{-1}(V) is equal to f^{-1}(V) restricted to j(X), which is open. Thus if p_0 does not belong to V then f ^{-1}(V) is open. So now let us consider the situation if p_0 belongs to V, then (X hat)\V is compact, and once again contained in i(X). So once again, f^{-1} of this, since it is contained in i(X) is equal to f ^{-1}((X hat)\V) restricted to j(X). But this is a compact subset of i(X), and this set is in fact equal to, f restricted to j(X) is equal to ioj ^{-1}, and both these are homeomorphisms.

This is from j(X) to X j^{-1} and this is i(X) right and both these are homeomorphisms. So this implies that f restricted to j of is a homeomorphism, which implies that f^{-1} restricted to j(X) is a homeomorphism, is continuous, we just need continuity. So this implies that f^{-1} restricted to j(X) of (X hat)\V is a compact subspace of j(X), and so also of T. This shows that f^{-1}((X hat)\V) is a compact subspace of T, and so also a closed subspace of T. So this implies that, but f((X hat)\V) is equal to T\f ^{-1}(V), this implies that f^{-1}(V) is an open subspace of T.

So this proves that f is continuous. This completes the proof. So we will end this lecture here.