Point Set Topology
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Lecture 31

Let us begin with the main theorem which we are going to prove. Let X be locally compact topological space which is not compact. Then there is no compactification. Then there is another topological space which we call X hat along with an inclusion i from X to X hat, such that the following conditions hold: First is (X hat)\i(X) is just one point which we denote p_0, Second, i is continuous. Third, the image of X under i is an open subset of X hat. Fourth is i(X) is dense.

Let us prove the existence of such a topological space X hat. So, the first condition already tells us that, so, we want i to be an inclusion, and (X hat)\i(X) is just one point. So, therefore, we have to take X hat is X disjoint union {p_0}. Next, we have to define a topology on X hat.

So, we define a collection $\mathcal{T}$ of subsets of X hat satisfying any one of the following two conditions: So, if U is an open subset of X, and U is open, and, if p_0 belongs to U and (X hat)\U, note that since p_0 is in U, (X hat)\U is a subset of X, which is contained in X, is compact in the subspace of X. So, we let $\mathcal{T}$ be the collection of subsets which satisfy these conditions and we need to check that $\mathcal{T}$ satisfies the conditions for being a topology on X hat. So, let us check the first condition. We need to show that the empty set and the full set X are in $\mathcal{T}$. So, the empty set satisfies (A), and the full set X hat satisfies (B).

Thus ϕ and X hat are in $\mathcal{T}$. Next let U_i be an arbitrary collection in $\mathcal{T}$. So, we need to show that V, which is the union of these is in $\mathcal{T}$, so, if all the U_i's satisfy condition (A), then clearly V, which is the union, also satisfies (A). Which would mean that each of the U_i's is contained inside X and each of these is open, which means the arbitrary union is going to be open. So, V also satisfies condition (A).

So, let us assume that one of these, Assume that, let us say U_{i_0} does not satisfy (A). So, it satisfies (B) for some i_0. Then, consider the set. Then V contains p_0 and therefore, to show that V is in this collection, we can only show that V satisfies (B). Consider the set (X hat)\V.

So, we are first look at the set (X hat)\V, this is equal to (X hat)\(union i in I of U_i's), which is equal to the intersection i in I of (X hat)\U_i's. As U_{i_0} contains p_0, this implies (X hat)\U_{i_0} is completely contained inside X. Thus we can write (X hat)\V is equal to intersection i in I of (X hat)\(U_i) intersected with X. So, if U_j satisfies (A), then this clearly implies that U_j is contained in X, and therefore, (X hat)\U_j intersection X is simply equal to X\U_j. This is a easy check and similarly, therefore, we can divide this into two parts intersection i in I U_i's satisfying (A), (X hat)\U_i intersection X, then this whole thing intersection with intersection i in I U_i's satisfies (B) (X hat)\U_i's.

Because of what we saw here, this is equal to intersection i in I U_i satisfies (A) X\U_i intersected with i in I U_i satisfies (B) (X hat)\U_i. So, this collection is non empty as U_{i_0} is here. Now, all these are closed in X and each of these is a compact subspace of X. This implies that (X hat)\V is a closed subspace of a compact subspace of X, which implies that (X hat)\V is a compact subspace of X right. This implies that V satisfies (B), which implies that V is in $\mathcal{T}$.

Therefore, $\mathcal{T}$ satisfies the second condition for being a topology. Finally, let us check that $\mathcal{T}$ satisfies the third condition for being a topology. For that let U_1,U_2 upto U_n be finitely many elements of $\mathcal{T}$, and we need to check that V which is the intersection of the U_i's is in $\mathcal{T}$. First assume that, consider the case where all U_i's satisfy (B). Then V contains p_0.

So, (B) was p_0 is in each of the U_i's, and X\U_i is compact. Then V contains p_0. So, (X hat)\V is equal to a finite union of (X hat)\U_i, each of these is compact, and clearly, a finite union of compact subspaces is compact. So, this implies that V satisfies (B), which implies that B is in $\mathcal{T}$. Now let us consider the second case which is: Assume that one of the U_i's satisfies (A).

So, then V does not contain p_0, and we need to show that V satisfies (A). That is the only way we can show that V is in $\mathcal{T}$. So, notice that as V does not contain p_0, it is already contained in X. Note that V is contained in X, as p_0 is not in V. Thus we can write V as V intersected X, which is equal to intersection i = 1 to n U_i intersection X.

If U_j satisfies (B), then it is easy to check, this is always true: U_j is equal to (X hat)\((X hat)\U_j). So, intersecting both sides, this is always true, this is a very simple set theoretic statement, since U_j is contained in X hat. Intersecting both sides with X, this implies that U_j intersection X is equal to X\((X hat)\U_j), but (X hat)\U_j is a compact subspace of X\U_j. satisfies (B), this implies a closed subspace of X. This implies X\((X hat)\U_j) is an open subspace.

Thus we get that V is equal to intersection i = 1 to n U_i intersected X. So, we can take this intersection into two parts: U_i satisfies (A), U_i intersected X intersected U_i satisfies (B) U_i intersected X. In both cases, each of these members is open in X. And since a finite intersection of open sets is open, this implies V is open in X. Therefore, we have checked that $\mathcal{T}$ satisfies all three conditions.

Thus $\mathcal{T}$ defines a topology on X hat. Next let us prove that, I should have said Hausdorff. Hausdorff, which is what we are going to check next. We have defined a topology on X hat. Let us check that this topology is Hausdorff.

If x,y are in X. We have to take any two points in X hat, and we have to construct neighborhoods of these points which are disjoint. If x and y are in X right then we are done, as X is Hausdorff. So, we can since X is Hausdorff, there are two neighborhoods, there are two open subsets U and V inside X, such that U contains x, and V contains y and their intersection is empty, and since both these U and V are in $\mathcal{T}$ by condition 1, therefore in this case we are done. Let us consider the next case, if x belongs to X and y=p_0. This is the only other case we need to consider.

Then let W be an open subset of x in X, such that the closure of W in X is compact. Such a W exists because we are assuming that X is locally compact. So, then (X hat)\W closure satisfies condition (B), and so is an open subset of X hat. So, clearly W and (X hat)\W closure are disjoint. Thus we have found a open neighborhood of X hat, an open subset of X hat containing p_0, another open subset of X hat which contains W, such that their intersection is empty.

This shows that X hat is Hausdorff. Next let us prove that X hat is compact. Suppose we are given an open cover. So, X hat is equal to union U_i's. So, there is a U_{i_0} such that p_0 belongs to this U_{i_0} ok.

So, thus (X hat)\U_{i_0} is contained in X and is compact. Now, so (X hat)\U_{i_0} is contained in this union U_i's and this implies that (X hat)\U_{i_0} intersected X is contained in this i in I U_i intersection X. Now, exactly as we saw over here, if U_i satisfies (B) then U_i intersected X is an open subspace of X. We saw this above right. This is if U_i satisfies (A) then obviously U_i intersected X is equal to U_i and it is an open subset of X and on the other hand if U_i satisfies (B) then as we saw above U_i intersected X is an open subset of X.

But this is equal to (X hat)\ U_{i_0} since (X hat)\U_{i_0} is contained in X. Therefore since (X hat)\ U_{i_0} is compact and it is a subspace of X, and we have found an open cover of this inside X, this open cover has a finite subcover. Thus (X hat)\U _{i_0} is contained in finite U_{i_j}'s, this implies X hat is contained in union of U_{i_j}'s with U_{i_0}. This proves that X hat is compact. Next we have the obvious inclusion.

So, if U contained in X hat is open and U is of type (A), then clearly i^{-1}(U) is simply equal to U, which is open. On the other hand if U satisfies (B) then as we saw above U intersection X is open in X. So, this implies the i inverse of U which is exactly U intersection X is open. Thus in both cases, the inverse image of an open subset is open.

This implies i is continuous. As we already seen that X hat is Hausdorff, and in a Hausdorff topological space, a single point is always closed, because the complement is an open subset. This implies that i(X) which is equal to (X hat)\{p_0} this closed subset, is an open subset. And the last thing we need to show is that i(X) is dense in X hat. For this, let U be an open set in X hat which contains p_0. So, we need to show that U intersection X is nonempty.

So, if U intersection X is empty. this will imply, as X is equal to (X hat)\{p_0}, is equal to X disjoint union {p_0}, this will imply that U is equal to p_0. That is the singleton set {p_0} is open in X hat. Thus, this will imply that X which is equal to X hat minus this open subset contains X hat, is a closed subspace which will imply that X is compact. But this but this contradicts our assumption. Thus U intersection X is nonempty, which implies that i(X) is dense in X hat.

This completes the proof of the theorem. So, we will end this lecture here.