Point Set Topology
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Week 01
Lecture 03

So let us continue with what we were doing. So this is lecture 3. So, we defined, so recall that $\mathcal{T}$ is a subset of the power set $P(\mathbb{R}^2)$ of $\mathbb{R}^2$ consisting of subsets U which satisfy this property (*) Recall what property (*) was. If x is in U, then there is $\varepsilon > 0$ such that $S_\varepsilon(a,b)$, so instead of x, I should write (a,b) is a point in U, is completely contained in U. So, let us check that $\mathcal{T}$ satisfies the conditions defining a topology. So, the first condition is ϕ and $\mathbb{R}^2$ should be in $\mathcal{T}$.

So, as in the example of the standard topology on $\mathbb{R}$, so ϕ is in $\mathcal{T}$ is vacuously true because there are no points in the empty set. Similarly $\mathbb{R}^2$ is in $\mathcal{T}$ is also clear, because if we take any point in $\mathbb{R}^2$, we can always take a square of side length 2. For any (a,b) in $\mathbb{R}^2$, we can take $S_1(a,b)$, and this is obviously contained in $\mathbb{R}^2$. So, obviously the first condition is satisfied.

So, let us look at the second condition. The second condition requires that if you take finitely many elements in $\mathcal{T}$, then the intersection of U_i's is in $\mathcal{T}$. So, to check this, once again, we take a point (a,b) which is in this intersection of U_i's and the same proof as in $\mathbb{R}$ holds. Since (a,b) is in U_i and U_i satisfies (*), there is $\varepsilon_i>0$ such that this $S_{\varepsilon_i}(a,b)$ is contained in U_i. So, we have our U_i, it may be some set like this, and we can take a point (a,b) here and there is some ε_i such that $S_{\varepsilon_i}(a,b)$ is in U_i.

So, we take ε to be equal to the minimum of $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ and clearly $S_\varepsilon(a,b)$ is contained in $S_{\varepsilon_i}(a,b)$. So, therefore $S_\varepsilon(a,b)$ is contained in $S_{\varepsilon_i}(a,b)$ which is contained in U_i for all i.

So, this shows that $S_\varepsilon(a,b)$ is contained in the intersection. So, thus the intersection satisfies this property (*) and so therefore the intersection of U_i's is in $\mathcal{T}$. So, therefore $\mathcal{T}$ satisfies condition 2. Let us quickly check that it also satisfies condition 3. So, condition 3 was given a set I and for each i in I, (we have) an element U_i in $\mathcal{T}$, and then we need to check that the union of U_i's is also in $\mathcal{T}$.

So, once again we let (a,b), we need to check that the union satisfies this property (*). So, let (a,b) be an element in the union which implies that (a,b) is in U_j for some j in I, which implies that there is $\varepsilon>0$ such that this square of side length ε, around (a,b) is contained in U_j as U_j is in $\mathcal{T}$ and so satisfies this property (*) and which implies that $S_\varepsilon(a,b)$ is contained in U_j which in turn is contained in this union. So, thus the union satisfies property (*) and so is in $\mathcal{T}$. So, this shows that $\mathcal{T}$ also satisfies the third condition to define topology. So, therefore $\mathcal{T}$ defines a topology on $\mathbb{R}^2$.

Our sixth example, which I will leave it as an exercise, is the standard topology on $\mathbb{R}^n$. So, this is very similar and let me just give some hints how to do how to define this. So, let us take a vector x_ in $\mathbb{R}^n$ (let us write a vector in $\mathbb{R}^n$ as x_). So, this is a n-tuple and we define, once again this subset, S_ε(x_) to be those y in $\mathbb{R}^n$ such that the absolute value of $|x_i - y_i| < \varepsilon$. Once again, we define this property (*) in the same way.

So, let $U \subseteq \mathbb{R}^n$ be a subset. We say U satisfies property (*): for every vector x_ in U there exists $\varepsilon>0$ (which once again, ε may depend on x_) such that this subset S_ε(x_) is contained in U. We let $\mathcal{T}$ be the collection of subsets U contained in $\mathbb{R}^n$, which satisfy property (*). And I will leave it as an exercise that show that $\mathcal{T}$ defines a topology on $\mathbb{R}^n$. This is an exercise, and so this is one exercise and I want to give another exercise.

So, before I write the exercise, let me just So in the example of $\mathbb{R}^2$ or even $\mathbb{R}^n$. So, in the example of $\mathbb{R}^n$ we can first define. So, let us write like this. So, define for x_ $\in \mathbb{R}^n$ and $\varepsilon>0$ define the subset B_ε(x_) as those y in $\mathbb{R}^n$ such that summation i = 1,2, .

.., n of $(y_i - x_i)^2 < \varepsilon$. So, this is the open ball of radius ε around x_. We can define a property (*'). Let U be a subset of $\mathbb{R}^n$. We say that U satisfies property (*') if for every x_ in U there is an $\varepsilon>0$, which may once again depend on this point x_, such that B_ε(x_) is contained in U.

and we let $\mathcal{T}'$ be the collection of subsets U such that U satisfies property (*'). So, then there are two exercises here. One, show that $\mathcal{T}'$ defines a topology on $\mathbb{R}^n$, and two, show that $\mathcal{T}$ (the topology in example 6), So, example 6 is this one. So, the standard topology on $\mathbb{R}^n$ that we define is equal to $\mathcal{T}'$. So, what I mean by the second exercise is: So, both $\mathcal{T}$ and $\mathcal{T}'$ are subsets of $P(\mathbb{R}^n)$.

Show that these subsets are the same. I am sorry, they are subsets of $P(\mathbb{R}^n)$. Show that $\mathcal{T}$ is equal to $\mathcal{T}'$, so both these subsets are the same. So, this the second exercise. Let me make a remark: exercise (2) above shows that the same topology can be defined in different ways.

So, we have seen quite a few examples of topological spaces now and we want to give more examples. So, we want to explain some tools, some methods, which we can use to define topologies on topological spaces. So, for that. So, next we want to explain some ways we can use, or some methods we can use, to define topologies on sets. So, our aim is to construct a large collection of topological spaces and study their topological properties.

So, first we have to construct a large collection of topological spaces. So, we are headed in that direction. Let us give a definition before we proceed, a very basic definition of open sets. So, let (X,$\mathcal{T}$) be a topological space. So, a subset U of X is said to be open in $\mathcal{T}$, if U belongs to $\mathcal{T}$.

So, if we have a set X, and we have given a topology $\mathcal{T}$ to X. So, subsets of X which are in $\mathcal{T}$ will often be called open subsets. So, whether U is open or not depends on the topology we have given to X. With this. So, let us define a basis for a topology.

Let X be a set and let $\mathcal{T}$ be a topology on X. A collection $\mathcal{B}$. So, $\mathcal{B}$ is a subset of $\mathcal{T}$, is called a basis for $\mathcal{T}$ if it satisfies the following condition: So, given any U in $\mathcal{T}$, and an x in U, there is an element W in $\mathcal{B}$, such that x belongs to W, and W is contained in U. So, let us see an example of a basis. We have already seen it in a form, but let us just make it more explicit.

So, example let X be the real line and let $\mathcal{T}$ be the standard topology on $\mathbb{R}$. So, let $\mathcal{B}$ be the collection of intervals (a,b) So, first of all we claim that every interval (a,b) is open in the standard topology on $\mathbb{R}$. What do we mean by this statement? We simply mean that every interval (a,b) is in $\mathcal{T}$. So, this is an easy check. We already did it in the example of (0,1).

So, you can try this as an exercise. It is the same. We showed that the interval (0,1) satisfies property (*) and in the same way. Hint: use the same method we used to show that (0,1) satisfies property (*) to show that the interval (a,b) satisfies property (*). So, recall that the standard topology on the real line was defined using the property (*) and we showed that the open interval (0,1) satisfies this property (*).

So, the same proof will can be easily modified to show that this interval (a,b) also satisfies property (*). So, the conclusion is that $\mathcal{B}$ is indeed a subset of $\mathcal{T}$, and we want to check that $\mathcal{B}$ satisfies this condition. So, let us check that. We claim that $\mathcal{B}$ is a basis for $\mathcal{T}$.

Let us prove this claim. Let U be in $\mathcal{T}$, then U satisfies property (*), which implies that for any x in U, there exists ε>0 such that (x-ε,x+ε) is contained in U. So, let W be the interval (x-ε,x+ε) Then x belongs to W, W is an interval and therefore it is an element of the set $\mathcal{B}$, and clearly W is contained in U. So, this shows that $\mathcal{B}$ is a basis for $\mathcal{T}$. So similarly, the collection Let me call it $\mathcal{B}$_2, consisting of those sets S_ε(a,b), where (a,b) is a point in $\mathbb{R}$^2 and ε>0, form forms a basis for the standard topology on $\mathbb{R}$^2. So, this can be checked in the same way that we check for ℝ, and once again similarly the collection $\mathcal{B}$_n, this is the collection of those S_ε(x_), where x_ is an element of ℝ^n and ε>0, forms a basis for the standard topology on ℝ^n.

So we have defined the basis for a topology, and you can familiarize yourself with the definition of a basis and these three examples, how these three examples of standard topology on ℝ and ℝ^2 and ℝ^n and the basis for these. In the next lecture, we will see two important properties that these basis have, that a basis has, and using those two properties, we shall explain how, given any subset $\mathcal{B}$ of the power set which has those two properties, we can use that set $\mathcal{B}$ to define a topology. So, we will end this lecture here. Thank you.