Point Set Topology

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Week 06

Lecture 29

In the previous lecture, we gave a criterion for metric space, of when a metric space is compact. So this lecture, we will begin with a lemma which is very useful, and it is called the Lebesgue number lemma, but before that we need some preliminaries. So, given the subset Z, we can define, so we define the function d_Z from X to $\mathbb{R}$ by d_Z(x) is defined to be infimum (all z in Z) of d(x,z). So, what is this mean? This is our x and this is my Z, let us say, so this is x. So, we take the distance of x from all points in Z and take the infimum. The infimum may or may not be attained at a point.

For instance, if we take the real line and we take the open interval (0,1), we take this to be Z and we take x to be -1. Then the infimum, so the distance between d(x,a)=a+1, if a is a point here, then this distance is a+1. The infimum is never attained because, I am sorry, so I should write it down. So, the infimum is 1, but it is not attained for any a in the subset Z.

The infimum may or may not be attained. So, this is a function which we define on X. Let us check that this is continuous. We claim that d_Z is a continuous function. For this note that if distance between x and y is strictly less than δ, then for any z in Z, we have the distance of x from Z is less than equal to X and Y, then Y and Z, which is strictly less than δ+d(y,z).

So, when we take infimum, we get d_Z(x) <= δ+d_Z(y), which implies d_Z(x)-d_Z(y)<=δ. Similarly, switching the roles of x and y, we get d_Z(y)-d_Z(x) is less than equal to δ, which implies the absolute value of d_Z(y)-d_Z(x) is less than equal to δ. So thus, for any ε positive, we let δ to be equal to ε/2. Then d(x,y) is less than equal to δ will imply that d_Z, let us say strictly less than ε, d_Z(y)-d_Z(x) is strictly less than ε. So, this shows that d_Z is continuous.

Using this distance function, let us prove the Lebesgue number lemma. Let X be a compact metric space, and let X, suppose we have given an open cover U_i's. So, then there exists δ positive, such that for every x, B(δ,x) is contained in U_i for some i. So now, this is something which we have already proved while we were proving our previous theorem because since X is compact, every sequence has a convergent subsequence and this is precisely the claim (1) which we proved while we were proving the previous theorem. However, using this distance function, let us see a clean/nice proof of this.

So, as X is compact, there is a finite subcover and we may assume X is equal to union i= 1 to n of U_i's. Let C_i be $X\setminus U_i$. These are closed subspaces of X. So, define a function f from X to $\mathbb{R}(x>=0)$ by f(x) is defined to be summation of i=1 to n of $d_{(C_i)}(x)$. So, given any point X, we take the distance of X from each C_i and we just add all these.

So, then f is continuous as all the $d_{(C_i)}$ are continuous. We are simply taking a finite sum of continuous functions and that is going to be continuous. If this function takes the value 0, then we have $\Sigma d_{(C_i)}(x)=0$. But each of these $d_{(C_i)}$'s is greater than equal to 0. This implies $d_{C_i}(x)$ is equal to 0 for all i, But what does that mean? So, note that recall that $d_{C_i}(x)$ is equal to infimum of z in C_i distance of x from Z, and this is equal to 0.

So, this implies that there is a sequence z_n in C_i such that the distance of x from z_n is tending to 0, but this implies that z_n is conveying to X. However, we also know that C_i's are closed, as Ci is closed in X. This implies that x belongs to C_i, which implies that the distance This implies that X belongs to C_i, that is all that we need, for all i. This implies that x belongs to intersection of C_i's. However, intersection of C_i's is equal to intersection of $X\setminus U_i$'s, which is equal to $X\setminus$(union of U_i's), but this is the empty set as the union is equal to X.

Thus we get a contradiction. Thus f(x) is always positive. So as X is compact, this implies f(X) is compact. This implies f(X) is closed in $\mathbb{R}$ and as 0 does not belong to f(X). As f(X) is closed and 0 is in the complement of f(X), which is open, this means there is a δ neighborhood of 0, which does not meet f(X).

This implies that f(x) is greater than or equal to δ for all x in X. So, this implies that the summation i=1 to n of $d_{C_i}(x)$ is greater than or equal to δ for all x. So, this implies that $d_{C_i}(x)$ is greater than or equal to δ/n for atleast one i. Because if it was strictly less than δ/n for all i, that would mean that the sum would be strictly less than δ. But what does this mean? So, we have our C_i, and our X is this.

So, distance of x from C_i is greater than or equal to, so this distance is greater than or equal to δ/n. But then the complement of C_i is U_i, so this implies that $B(\delta/n,x)$ is contained in U_i. So, this proves the lemma. So, we will end this lecture here.