Point Set Topology
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Lecture 26

The main theorem that we proved in the previous lecture was that to check if a subspace of ℝ^n is compact, it's enough to check that the subspace is closed and bounded. As an application of this we proved that SO(n) is compact. In the previous lecture we proved that our subspace Y of ℝ^n is compact if and only if Y is closed and bounded. As an application we saw that SO(n) is compact. Similarly, we can prove that O(n), U(n), SU(n) are compact. Slight modifications of the proof that SO(n) is compact will show us that O(n), the group of unitary matrices, and special unitary matrices, these are all compact.

So, today we will in this lecture we will first study how compactness behaves with respect to continuous maps. So, let us begin with this proposition. Let me just write that. So, behavior of compactness with respect to continuous maps.

So, we begin with this proposition. Let f from X to Y be a continuous map. The image of a compact set of compact subset is compact. Let us prove this. So, let Z contained in X be compact.

Then we need to show that f(Z) is compact. To show that f(Z), or rather a subspace is compact, it is enough to show that if Y' is contained in a cover U_i, where U_i is open in Y, then Y' is contained in a finite subcover. So, we have seen this already on two occasions before. So, let us just prove this, it is easy. So, suppose Y' is compact and we are given that Y' is contained in this cover U_i.

This implies that Y' is equal to, this is an open cover for Y'. And since Y' is compact, this implies that Y' is equal to, there is a finite indexing set U_ij, which implies that Y' is contained in union of U_ij. Conversely suppose Y' has this property: suppose Y' has the above property, has the property that if we put Y', if Y' is contained in an open cover, where U_i's are open subsets of Y and not of Y' then it is contained in a finite subcover. Then we show that Y' is compact as follows. First let us take Y' be equal to union i in I of W_i's, be an open cover of Y'.

Now Y' has a subspace topology which means that each W_i is equal to Y' intersection U_i for some U_i open in Y. So, this implies that Y' is contained in union i in I of U_i, which implies that it is contained in, because it has this property U_ij, which implies that Y' is equal to union j=1 to n of W_j's. This implies Y' is compact. We will repeatedly use this remark. So, if you want to show that a subspace is compact, we will show that every time it is contained in an open cover, it is contained in a finite subcover of that open cover.

So, we need, in our case we need to show that f(Z) is compact. Suppose f(Z) is contained in the union of, let us say V_i's, where V_i is open in Y. So, then we have Z is contained in this cover i in I $f^{-1}(V_i)$'s, and since f is continuous, this is open in X, and as Z is compact there is a finite set, so Z is contained in union j=1 to n of $f^{-1}(V_j)$, which implies f(Z) is contained in union j=1 to n of V_j, thus f(Z) is compact. So, this completes the proposition, and we will prove the following lemma. So, the following proposition is very useful.

Let f from X to Y be a bijective continuous map. If X is compact, then f is a homeomorphism. Proof: so to show that f is a homeomorphism it is enough to show, since f is bijective and continuous, it is enough to show that the image of an open subset is open and which happens if and only if the image (that is an easy check and I will give it to you as an exercise) of a closed subset is closed. So, we will show this. Let Z contained in X be a closed subset, then as X is compact, this implies Z is compact.

This is because we have proved that the closed subspace of a compact space is compact, which implies that f(Z) is compact, which implies that, once again we are under the hypothesis that all our spaces are Hausdorff. So, in a Hausdorff space a compact subspace is closed. So, f is a bijective continuous map and it takes closed sets to closed sets. From this we can easily conclude that it takes open sets to open sets. Or in any case, from this we easily conclude that f is a homeomorphism.

Before we start a discussion on, this was a very general discussion on compactness, and before we start a discussion on compact metric spaces. So, just as earlier, in case of metric spaces we had given a special description of closed subspaces, we had explained how to describe closed subspaces and how to describe continuous maps between metric spaces. So, in case of compactness also, we can make some comments when our space is a metric space, but before we go on to that we will state this following very important theorem which we will not prove. Let X_i be compact topological spaces for i in I. So, this capital I is now an infinite, possibly infinite index set, then this product of X_i's, obviously this is given the product topology, is compact.

So, this is called Tychnoff's theorem and we will not prove this in this course. A proof may be found in Munkres. So, we will end this lecture here.