Point Set Topology
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Week 05
Lecture 24

In the previous lecture, we introduced compactness and we saw that the closed interval [0,1] is compact. So, just as in the case of connectedness, we can ask what happens for products or subspaces. So, if X is compact, so what is compactness behave with taking products or taking subspaces. So, exactly as in the connected case, if X is connected and Y is a subspace, then Y need not be connected. For instance, we can take X equal to ℝ and Y we can just take {0,1}. Clearly Y is just the union of these two points.

So, clearly Y is not connected and in the same way, if X is compact, and Y contained in X is a subspace, then Y need not be compact. So, we can take X equal to [0,1] and Y equal to the open interval (0,1). So, there is a homeomorphism from ℝ to (0,1) and we saw that ℝ is not compact. Thus, the open interval (0,1), since homeomorphism obviously is going to preserve compactness, isn't compact.

So, we saw that ℝ is not compact, we explicitly wrote down an open cover which does not have a finite sub cover. What this homeomorphism is, is left as an exercise. So, let us come to products and that is as in connectedness, if X and Y are connected topological spaces, we saw that their product is connected. In the same way we will prove that if X and Y are compact topological spaces, then their product is also compact. So, the crucial ingredient for that we need is this lemma which is called "tube lemma".

Let W contained in XxY, so XxY is obviously always given the product topology, be an open set contained in this XxY. So, then there is an open set U contained in X, such that x is in U and Ux{y} is contained in W. So, let us make a picture of this lemma and explain what it is saying and why it is called the tube lemma. So, this is our X and this is our Y which is compact and here we have this is XxY and we have this open set W, W is some open set which contains {x}×Y. What this lemma says is there is a small neighborhood U around x such that this entire disk around {x}xY is contained inside W.

It contains this entire tube W, so this region is W and this is UxY. So, let us see how to prove this lemma. So, given any point y in Y there is a basic open set. So, given any point in Y, we have the point (x,y) which is in XxY which is contained inside W and W is open in the product topology, which implies there exists open sets U_y. contained in X and V_y contain in Y such that this (x,y) this is in U_y x V_y.

and this is completely contained inside W. So, we are taking any point over here this point is (x,y). So, we are just saying that there is open neighborhood. This is U_y and this is V_y. So, U_yxV_y is contained in W.

Clearly and this happens for every y. Thus we can write Y as a union of y in Y of V_y. Since Y is compact this has a finite sub cover. So, we can write y as j=1 to n of V_{y_j}, we can write this as a finite union. Let U be equal to intersection of the U_{y_j}'s.

Note that each of these U_{y_j}'s contains x. So, therefore, U contains x, each of these U_{y_j}'s is a open set containing x, we are taking a finite intersection. Therefore, U is an open set of X, containing the point x. Then UxV_{y_i} is contained in U_{y_i}xV_{y_i}, which is contained in W and this happens for all i. This implies that union i =1 to n, U x V_{y_i} is contained in W, but this union is simply U x union V_{y_i}'s is contained in W, but the union of the V_{y_i}'s is simply Y is contained in W.

So, this completes the proof of the law. So, using this lemma which is the crucial ingredient let us prove the theorem. Theorem: let X and Y be compact topological spaces, then the product XxY is compact. Let us prove this: suppose we are given an open cover. So, then fix an x in X.

Then we have {x}xY which is contain in XxY = union of W_i's. So, as x x y. So, note that Consider the map Y to XxY which sense y to (x,y). So, this map is bijection to XxY and this has this subspace topology. So, it is easy to check.

The map from y to {x}xY is obviously continuous because the image of Y, let us call this f_0 is a homomorphism. So, this shows that {x}xY is homomorphic. This {x}xY is compact and now, we can write {x}xY is equal to union of ({x}xY) intersection W_i's and since {x}xY is compact, this implies that this has a finite subcover which implies that this {x}xY is actually contained in some finite set W_{i_j}. By the previous lemma, let us write this finite subcover as union of all i contained in I_x of W_{i}. So, {x}xY is contained in some finite subcover, we just collect the indices in that subcover and put it into the set I_x.

So, here this is a finite set. Now by the previous lemma there is an open neighborhood U_x of x in X such that, What is the previous lemma say? So, let us look at the previous lemma. Y is compact, and if {x}xY is contained in W, then there is a small neighborhood U_x such that UxY is also contained in W. So, we will use that {U_x}xY is contained in union W_i. Now, this can be done for every x in X.

So, thus when we do this, we get an open cover of X, U_x. Now, again as X is compact, this has a finite subcover. We can write X as union U_{x_j}. So, then x is equal to, I am sorry XxY = union of {U_{x_j}}xY which is contained in union of W_i's, each of these i's is contained in this indexing set. And now the index set is finite here.

So, as each I_{x_j} is a finite set, subset of I this implies, this is equal to union of all i=1 to n of W_i's, this index set is finite. So, thus XxY, thus we have found a finite subcover for XxY. So, this sub cover is finite because this index set over here that is a finite index set. So, this completes the proof.

. As a corollary of this we see that the closed interval [0,1], I mean when we take product of it n times, this is compact. Next let us make some observations about compactness. Proposition: a closed subspace of a compact space is compact. So, let us prove this. Let Y contained in X be closed and suppose that X is compact.

Our aim is to show that Y is compact. What we have to show is, given any open cover for Y, it has a finite subcover. Given any open cover for Y, we will construct an open cover for X, and from that we will deduce that Y is compact. Let Y be equal to union of i in I, W_i, be an open cover for Y. Since y has a subspace topology, thus there exist U_i's contained in X, that are open, such that each W_i is equal to Y intersection U_i.

So, this implies that Y is contain in U_i's, in this collection of sets open in X. So, suppose this is our X, and let us say this is our Y. First, we are given an open cover for Y. We extend this to an open cover for X. I am sorry, we cover this like this, we find open subsets, we find a collection of open subsets in X, such that their union contains Y.

Now since Y is closed, we may write close in X, we have X\Y is open in X. So, this region is open. We can write X as X\Y disjoint union Y, this contains X\Y union U_i's. So, this is open in X, and each of these is open in X. So, X is contain in this, and all these are of course contained in X.

This implies that we have got an open cover for X. Since X is compact, this has a finite subcover. So, let us say the finite subcover looks like this, I mean this may or may not be present, but there is no harm in throwing in an extra open set in the open cover. So, now intersecting both sides with Y, we get Y is equal to: this (X\Y) is disjoint, this is empty, union Y intersected with W_{i_j}. Thus, the open cover we started with has a finite subcover.

This proves that Y is compact. So, we will end this lecture here and in the next lecture we shall show that a closed subspace of ℝ^n is compact if and only if it is closed and bounded.
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