Point Set Topology
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Lecture 23

So, welcome to this lecture. So, in the last few lectures we studied the topological property of being connected and path connected and saw several examples. So, two examples which we did not see were: The problems of whether SO_n, U_n are connected or path connected will be pending, They will be taken up later. We need to develop a few more tools before we can answer the question of whether SO_n and U_n are path connected. However for the next few lectures we will study the property of being compact. Let us begin with the definition of a Hausdorff topological space.

Let X be a topological space. We say that X is Hausdorff if for distinct points x_1 and x_2 in X, We have our X, we have x_1 here and we have x_2 here. So, there should be two small open sets which contain x_1 and x_2 which are disjoint. So, we can take any two distinct points x_1 and x_2 and there exists open sets U_1 and U_2 such that x_1 belongs to U_1, x_2 belongs to U_2 and U_1 intersection U_2 is empty.

If this happens for all pairs of distinct points x_1 and x_2, then we say X is Hausdorff. Here are some easy exercises. If X and Y are Hausdorff, then the product with the product topology is Hausdorff. So, let us see how to do this. Let us say we have two points (x_1,y_1) and (x_2,y_2).

So, now, since (x_1,y_1) is not equal to (x_2,y_2), this implies either y_1 is not equal to y_2 or x_1 is not equal to x_2. So, let us simply assume that y_1 is not equal to y_2. So, then our y_1 is here and y_2 is here. Since Y is Hausdorff, this implies there exists open sets such that y_1 belongs to V_1, y_2 belongs to V_2 and V_1 intersection V_2 is empty. So, this is our V_2 this could be our V_1.

So, then XxV_1 and XxV_2 are open sets in the product topology XxY. So, this is XxV_2. Clearly (x_1,y_1) is in XxV_1, (x_2,y_2) is in XxV_2 and the intersection is empty. This shows that XxY is Hausdorff. In the same way we can easily check that if X_i is a family of Hausdorff topological spaces, then the product (is Hausdorff).

Obviously every time we talk of the product as I said, we are giving it the product topology, and then imitate the proof of 1. The same proof as in the previous case will work and I will leave it as an exercise. Finally the third easy example/exercise is: X is Hausdorff and Y contained in X is a subspace, then Y is Hausdorff. Let us quickly see this. So, suppose this is X and let us say Y is this.

So, if you take any y_1 and y_2, then since X is Hausdorff, there is a open set U_1 containing y_1, U_2 containing y_2, such that U_1 intersection U_2 is empty. Then clearly U_1 intersection Y contains y_1, U_2 intersection Y contains y_2, and these are disjoint open subsets of Y, thus Y is Hausdorff. The above exercises give us plenty of examples of Hausdorff topological spaces. We are mostly interested in Hausdorff topological spaces, and we will almost deal exclusively with Hausdorff topological spaces for the rest of this course. So, for the rest of this course we will be interested only in Hausdorff topological spaces.

If nothing is mentioned then it is safe to assume that the topological space we are working with is Hausdorff. We have defined what Hausdorff means. Now we are ready to define compactness. Let X be a Hausdorff topological space. We shall say that this is compact if the following happens: So, every open cover U_i, i in I.

What do we mean by an open cover? It means we are given a family of open sets U_i, and the union of all of them is equal to X. We are given any open cover and given any open cover has a finite subcover, that is, there exist finitely many indices i_1, i_2 upto i_n such that X can actually be covered by these finitely many open subsets U_{i_j}. Let us see an example of a topological space which is not Hausdorff, I am sorry which is not compact. Example: $\mathbb{R}$ is not compact. So, to say that $\mathbb{R}$ is not compact, it is enough to produce an open cover for $\mathbb{R}$ which has no finite sub cover.

So, that is easy. So, what we do is we take an n, n+1 and we take (n-1/4, n+1+1/4). Then we can write $\mathbb{R}$ as the union over n in integers of (n-1/4, n+1+1/4) So, clearly any finite sub collection in this union, it will not cover $\mathbb{R}$, any finite subcollection of intervals of the type (n-1/4, n+1+1/4) will not cover $\mathbb{R}$. So, thus this cover has no finite subcover, thus $\mathbb{R}$ is not compact. So, $\mathbb{R}$ is not compact.

So, now, we want to construct some very basic examples of compact spaces, and prove some very basic results about compact spaces which is exactly the same as we did when we talked about connectedness. The first result we are going to prove is the interval [0,1] is compact. The proof is very similar to how we proved that [0,1] is connected. So, suppose we are given an open cover, let U_i's be an open cover. So, our aim is to find a finite subcover of this.

We need to show that this has a finite subcover. As before we let S be equal to those x in [0,1] such that this when we look at this interval, the closed interval [0,x] is covered by finitely many U_i's. So, clearly 0 belongs to S. For 0 to belong to S what we need is [0,0], this is just $\{0\}$, is covered by one U_i. We can just take any U_{i_0} which contains 0 or any U_j, let us say any U_j which contains 0, and then obviously this interval which is just $\{0\}$ is contained in this U_j.

So, when I say covered by finitely many U_i's, I mean that this interval $[0,x]$ is contained in finitely many U_i's, union of finite union of U_i's. So, let x_0 be equal to supremum of x in S_x. So, we claim that x_0 is also in S. So, let us prove this. So, there is a sequence x_n's converging to x_0 with x_n's in S.

So, let x_0 be in some open set. So, we have 0, we have 1 and let us say our x_0 is here. So, x_0 is going to be in some open set U_{i_0}. So, there is an $\varepsilon>0$ such that $B_\varepsilon(x_0)$ intersected with $[0,1]$ contains U_{i_0}. And since these x_n's are converging to x_0, all but finitely many x_n's are in this interval.

So, all these x_n's eventually, all of them will land inside this small open neighborhood around x_0. So, we choose some x_n in $B_\varepsilon(x_0)$ intersected with $[0,1]$. This 0 is x_0, this are in this interval, and let us say x_m is here, let us say this is 1. So, then as x_m is in S, this implies $[0,x_m]$ is covered by or let us say is contained in a finite union of U_i's. And moreover we also know that this this interval $[x_m,x_0]$ is contained in $B_\varepsilon(x_0)$ intersected $[0,1]$ which is contained in this U_{i_0}.

So, this implies that $[0,x_0]$, which is equal to $[0,x_m]$ union $[x_m,x_0]$, which is contained in $[0,x_m]$union U_{i_0}. Now, this interval $[0,x_m]$ is contained in a finite union and therefore $[0,x_0]$ is contained in that finite union along with U_{i_0}, which is therefore also going to be finite. So, therefore we conclude that $[0,x_0]$ is contained in a finite union of the U_i's. Thus x_0 is in S. So, this proves our claim the claim that x_0 is in S.

So, now, next we claim that $x_0=1$. So, once again, if x_0 is strictly less than 1, then that would mean that there is some... So, let me just make a picture here once again, and this is our ε, and this is our x_n, that would mean that there is some $\varepsilon>0$, such that $[x_0,x_0+\varepsilon]$ is contained in U_{i_0}.

Again so this implies that $[0,x_0+\varepsilon]$ which we can write as $[0,x_m]$ union $[x_m,x_0+\varepsilon]$ is contained in a finite union of the U_i. So, this implies that $x_0+\varepsilon$ belongs to S, which contradicts the assumption that x_0 was the supremum. So, thus x_0 has to be 1, x_0 cannot be strictly less than 1. And since x_0 is equal to 1, this implies that $[0,1]$, we have already seen that x_0 is in S, therefore this implies that $[0,1]$ is contained in a finite union of the U_{i_j}. So, thus there is a finite subcover is contained in j equal to 1 to n some U_{i_j}.

So, therefore this open cover. So, this proves that the open cover we started with has a finite subcover, thus this interval is compact. So, this completes the proof of the theorem. So, we will end this lecture here.