Point Set Topology
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Lecture 21

In this lecture we shall sketch a proof of the fact that $GL_n(\mathbb{R})^+$, so this is those n x n matrices such that determinant of A is positive, is path connected. We shall see this as an application of what we have done so far and so let us denote by G this $GL_n(\mathbb{R})^+$. We will do this, first of all it suffices to show that any matrix A in G can be connected to the identity by a path in G. We have our G over here, given any matrix A we can connect it to identity using a path. That is what we are going to prove, and we will do it in several steps. So, let us begin, so let A be in G, so then the first step is then we can join A to B in G.

So, when we say join A to B in G, we always mean join by a path which is completely contained inside G. Where B is such that b_{11}, the first entry is not 0. So, how do we do this? If a_{11} is nonzero, then we just take to be the constant path. So, $\gamma(t)=A$, so the constant path is continuous and therefore we can just take B = A.

On the other hand, if $a_{11}=0$, then since G is an open subset of $M_n(\mathbb{R})$, and $M_n(\mathbb{R})$ has the product topology, there exists an $\varepsilon>0$ such that the set U is those B in $M_n(\mathbb{R})$ with $|a_{ij}|-|b_{ij}|$, is strictly less than ε for all i,j. This is going to be contained in $GL_n(\mathbb{R})$ simply because $GL_n(\mathbb{R})^+$ is open inside $M_n(\mathbb{R})$, and $M_n(\mathbb{R})$ has a standard topology or product topology. We can see this in many ways, so we just take any B in U with b_{11} not equal to 0. Then take the path γ from [0,1] to U, given by $\gamma(t)=tA+(1-t)B$. So, this path is going to be completely contained inside U, because each coordinate of $\gamma(t)_{ij}$ right, so this is a_{ij} and this is b_{ij}, so the distance between them is less than ε.

So, for any t, $\gamma(t)_{ij}$ is going to be over here, somewhere in between a_{ij} and b_{ij}, so the distance of that from a_{ij} is going to be less than ε. So, it is going to be contained inside U, which is contained inside G. So this gives the required step 1 right, so we have joined A using a path to this matrix B, and b_{11} is nonzero. So, let us go to step 2, if B is in G and b_{11} is nonzero, let so recall this E_{ij}, these are the elementary matrices, so these are matrices of the type: So, these are matrices of the type 1, these entries are something and on the diagonal, we have 1 and all these are 0. So, there is a matrix of this type such that E_1 when we multiply it on the left by B, all these, except for the first one, all the other entries in the first column becomes 0.

So, this first one is going to be λ, so let us assume that b_{11} is λ. So, similarly there exists a matrix E_2 right of the type 1, so the first row is of this type, and in the diagonal we have

1's, and all the other entries are 0, such that E_1*B*E_2 is of this type. So, λ, all the other entries in the first row are 0, all the other entries in the first column are 0, and here we have a matrix D. So, consider this map [0,1] to GL_n(ℝ)^+ This is given as follows: t goes to (1-t)I+t(E_1)*B*(1-t)I+t(E_2). So, we need to check that: First: as a map of sets, the image actually lands inside GL_n(ℝ)^+, but when we take determinant, for each t, let us look at this matrix (1-t)I+t(E_1) So, E_1 is a matrix of this type, so one easily checks that, once again, this is going to be a matrix of this type.

So, this implies determinant of this matrix is equal to 1. So, similarly determinant of (1-t)I+t(E_2) is going to be 1. so this matrix is now going to be of this type here: 0 and the first is going to be of the same type as E_2. So, this implies that determinant of γ(t) is actually equal to determinant of B, which we know is positive. Therefore, as a map of sets, the image of γ lies inside GL_n(ℝ)^+, next we want to check it is continuous here, but to check γ is continuous we can just, since GL_n(ℝ)^+ is contained in M_n(ℝ), and has a subspace topology it suffices to check that the coordinates of γ are continuous.

And the coordinates of γ are polynomials with these coordinates, are polynomials in t. and so are continuous, that is easily checked when we multiply out these matrices it is clear that the coordinates will be polynomials in t and So γ is continuous, since each coordinate function is continuous. So, we have this path γ from [0,1] to GL_n(ℝ)^+, now γ(0)=B, we can check it is equal to B, and γ(1) is equal to E_1*B*E_2, which we know is a matrix of this type. So, this completes step 2. So, if B is a matrix such that b_11 is nonzero, then B can be connected to a matrix of this type.

So, let us go to step 3. So, if λ is positive, then this matrix [[λ, 0],[0,D]], Let us look at this path, so consider C to be equal to [[λ, 0],[0,D]]. Then let λ be positive. So, then C can be joined to [[1, 0],[0,D']] So, obviously this is in G. So, we are starting with some matrix C in G, of this type, then we can join it to some D', where D' belongs to GL_{n-1}(ℝ)^+.

So, how do we do that? So we look at this path. We look at this matrix, [[t/λ +(1-t),0],[0,I]]. So, this is a diagonal matrix. So, in a_11, it has this entry, and in all the other diagonal entries it has 1. So, what happens over here at this entry? So, this is joining λ is positive, so 1/λ lies over here and let us say 1 is over here.

Then this is the straight line joining 1 with 1/λ, and we just multiply this with C. So, when we take determinant, so let us just once again, we check that when we take this is equal to γ(t). Determinant of γ(t) is equal to t/λ+(1-t)*det(C), which is positive. So, therefore the image of γ actually lands inside GL_n(ℝ)^+ and once again, this is continuous because each of the coordinate functions are polynomials in the variable t. And notice that γ(0)=C and γ(1) is [[1/λ,0],[0,I]]*C.

So, this is equal to [[1,0],[0,D]], this is in GL_{n-1}(ℝ)^+. So similarly, if λ is strictly less than 0, then this path [[t/|λ|I+(1-t),0],[0,I]]*C connects C with a matrix of this type. So, let us say some D'. So, now we want to show that this matrix D' can be connected to matrix of the form [[1,0],[0,D"]]. So, this is in G, and we want to connect these two, via a path in G.

So, for this, so consider this matrix: t[[-1,0],[0,-1]]+(1-t)[[0,1],[-1,0]], this 2x2 matrix, this is equal to [[-t,(1-t)],[(1-t),-t]] and this has determinant t^2+(1-t)^2, which is positive. Similarly, t times this matrix plus 1-t times [[0,1],[-1,0]] has determinant positive. So, what this means is that [[-1,0],[0,-1]] can be joined to I in GL_2(ℝ)^+, In fact, this first thing that is a path from [0,1], that is a path joining [[-1,0],[0,-1]] to [[0,1],[-1,0]], in this path is completely contained inside GL_2(ℝ)^+, And similarly, there is a path completely contained inside GL_2(ℝ)^+ which joins this matrix to [[1,0],[0,1]], the identity. Therefore this -I can be joined to I in GL_2(ℝ)^+ using a path inside GL_2(ℝ)^+. We will use this by the path.

So, there exist such a path because we can just put these two paths together as we had seen when we proved that being when we defined path equivalence classes. So, now we define a map from γ from [0,1] using this γ, let us say γ~, to GL_n(ℝ)^+ by [[γ_11(t) γ_12(t) 0],[γ_21(t) γ_22(t) 0][0,0,I]] So, clearly this path joins [[-1 0 0],[0 -1 0],[0 0 I]] to the identity in GL_n(ℝ)^+. So then the path from [0,1] to GL_n(ℝ)^+ given by t goes to ok. So, this path is what we are calling γ~(t). So, γ~(t)[[-1 0][0 D']].

So, it joins [[-1 0][0 D']] to [[1 0][0 D"]]. So, thus we conclude that if C in GL_n(ℝ)^+ is a block matrix is of this type here. Then C can be joined by a path in GL_n(ℝ)^+ to a matrix of the type [[1 0][0 D"]] and then D" is forced to be in GL_n(ℝ)^+. Now what have we done so far: we start with the matrix A we connected it by a path to a matrix B such that b_11 is not equal to 0, then we connected it to a matrix C such that C is of this type. So, then we connected this to a matrix of the type [[1 0][0 D"]] So, by induction on n, we may assume that D" can be connected to I in GL_{n-1}(ℝ)^+ using a path γ.

Then the path [[1,0][0, γ(t)]] in GL_n(ℝ)^+ connects D" to the I. We have this path is connected. Thus we have proved, this is only a sketch and I will leave it as an exercise to fill in the details and convince yourself that all the arguments are correct, every element A in GL_n(ℝ)^+ can be connected using a continuous path γ from [0,1] to GL_n(ℝ)^+ to the identity. So, this proves that GL_n(ℝ)^+ is connected, is path connected and in fact connected because we know that path connected spaces are connected. So we will end this lecture here.