Point Set Topology
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Lecture 20

So, welcome to this lecture. In the previous lecture we had introduced the notion of path connectedness, and we defined an equivalence relation on X, and we ended the lecture by stating this proposition. Let us quickly see a proof of this proposition, it is really easy. First we want to show that every path connected space of X is contained in some X_i. Let T be a path connected subspace of X, Let X be in T, then X belongs to X_i for some i. Now, we claim that T is contained in X_i.

To show that T is contained in X_i, it suffices to show that, for any t in T, there is a path in X joining x and t, but as T is path connected this implies that there is a continuous path in T such that $\gamma(0)=x$ and $\gamma(1)=t$. But then we can just take the inclusion in X, which is continuous and therefore the composite of these is going to be continuous. So, this implies that T and x are in same path component. So, this implies that T is contained in X_i.

This proves 1. We have shown that every path connected subspace of X is contained in some X_i. Part (2) is also obvious. Let x be in X_i. We need to show that any two points in X_i can be joined by a path.

This is clear since if x and y are any two points in X_i then this is clear from the definition of path components. Both these together prove that each X_i is a maximal path connected subspace of X. Recall that this is a similar proposition, when we were talking about connected components, but that proposition had third point, which said that every connected component is closed. That is not true of path components. So, let us see a counterexample.

A space which is connected but not path connected. First consider this: let us make a picture of this space. This is 1, and let us take 1/2, and this is 1/3, this is 1/4, So, at each of these 1/n's we make a straight line of length 1. We remove the origin. Let us just look at what the space C is.

C is a subset of $\mathbb{R}^2$, and it consists of points (1/n,y). So, in the x coordinate we have 1/n, where n is a natural number and y is in the interval [0,1]. So, C will contain all these lines. We also need to take this region. We delete the origin but we take the rest.

So, let me just write what this is. This is the points (0,x), the x coordinate is 0, and y coordinate is in the half open interval (0,1]. In particular this point P is (0,1). This is this line. We also have to take part of the x-axis, which is union (x,0).

The y coordinate is 0 over here, and where x, we have to remove the origin, and we can take any x. This region in black. We call them of the same region, this is our C. Let C have the subspace topology from $\mathbb{R}^2$. $\mathbb{R}^2$ has the standard topology, which is the same as the one given by the standard Euclidean metric.

Now, let γ from [0,1] to C be a path such that γ(0)=(0,1), which is this point P. Then the claim is the image of γ is completely contained inside Y. What is Y? Y is the subspace that is this line Y over here. The starting point of γ is this point P, and the claim is if you take any continuous map from [0,1] to C which starts at this point P, then it cannot go outside the subspace Y. Let us prove the claim.

Let us assume this is not true, and that the image of γ moves out of Y. Note that Y is a closed subspace of C. In fact, C is contained in $\mathbb{R}^2$ and it has a subspace topology from $\mathbb{R}^2$, we have the projection to Y. This is the projection to the second coordinate. And the subspace Y is exactly equal to Let us say this is i.

So, i composed p_2. So, p_2 is continuous, the projection maps are continuous, and the inclusion is continuous because C has subspace topology and therefore the composite is continuous, and Y is exactly the inverse image of 0 in C. Therefore Y is a closed subspace of C. Now we define this set S={x in [0,1] such that γ([0,x]) is completely inside Y}. Let t_0 be the supremum over S.

So, S is contained in [0,1] and clearly S is nonempty as 0 belongs to S. because γ(0), this interval [0,0] which is just γ(0), which is equal to P, this belongs to Y. Then let t_0 be the supremum over all s in S. We claim that first $γ(t_0)$ is in Y. So, why is that? Since t_0 is the supremum, this implies there is a sequence t_n in S, such that t_n's converge to t_0.

Now this as γ is continuous, this implies $γ(t_n)$ converges to $γ(t_0)$, but note that as t_n belongs to S, $γ([0,t_n])$ is contained in Y, which in particular implies that $γ(t_n)$ belongs to Y, and since Y is closed, every time we have a sequence of points inside Y which converges to some point, it will mean that that limit point is contained in Y. This implies that $γ(t_0)$ is in Y. If $t_0=1$ then what is this mean? Then this implies that γ([0,1]) is contained in Y, which is what we want to prove. So, then we are done. Let us assume that this is not the case.

Let us assume that $t_0<1$. We have this interval [0,1] and t_0 is somewhere over here. Let us just copy this diagram, and this is our γ. So, $γ(t_0)$ is contained in Y.

This point is $\gamma(t_0)$. We just showed that $\gamma(t_0)$ is contained in Y. So, this is our Y. We take a small neighbourhood U around $\gamma(t_0)$ such that U does not contain the origin. So U is, let us say a small neighborhood in $\mathbb{R}^2$. We could, for instance, take a small square like this.

So, then γ inverse, so γ is a map from [0,1] to C. which is contained in $\mathbb{R}^2$. So, let us just view γ as a map from [0,1] to $\mathbb{R}^2$, is therefore continuous, because C has a subspace topology, and therefore the inclusion is continuous. So, we can just take γ inverse of U. Viewing γ as a map from [0,1] to $\mathbb{R}^2$, is an open subset open in [0,1] and so it contains the interval $(t_0, t_0+\varepsilon)$ for some $\varepsilon>0$.

Now note that as t_0 is the supremum of elements in S, the image $\gamma([0,t_0])$ is contained in Y. Moreover we have proved that $\gamma(t_0)$ is also in Y. This implies that image of this closed interval is contained in Y. But now if we take any $\delta>t_0$, then $\gamma([0,\delta])$ cannot be contained in Y, otherwise t_0 will not be the supremum of S. Thus there exists δ in this half open interval such that $\gamma(\delta)$ does not belong to Y.

We restrict γ to $[t_0,\delta]$. So, this restriction is a mapping to C, but the image actually lands inside U intersection C, which is contained inside. Therefore we get that $[t_0,\delta]$ to U intersection C is continuous. Now let us make a picture of U intersection C. It looks something like this. So, this is U, this is $\gamma(t_0)$ and there are all these lines at 1/n, which are converging to x=0.

And $\gamma(\delta)$ is somewhere over here, so $\gamma(\delta)$ is not in Y, so it lands outside, and it also lands inside U. Therefore it is of the type (1/n,y) for some n. But now, we can find two open subsets, nonempty open subsets in U intersection V. So, let us say this is V_1 and this is V_2. Then U intersection C we can write it as V_1 intersected C disjoint union V_2 intersected C.

Now $\gamma([t_0,\delta])$ contains $\gamma(t_0)$ and this contains $\gamma(\delta)$. So, γ is from $[t_0,\delta]$ to U intersection C, is continuous. This implies that $[t_0,\delta]$ can be written as the disjoint union of γ inverse V_1 intersected C and γ inverse of V_2 intersected C. But this is a contradiction, as $[t_0,\delta]$ is connected. So this would contain t_0 and this would contain δ, so that is not possible.

Plus, t_0 has to be equal to 1, and this implies γ of this entire interval is contained in Y. What does this mean? So this means that if we take a path, which starts at P then it has to remain inside Y. It cannot move outside Y. So, if I take this point this is (1,1). This shows that there is no continuous path from (0,1) to C with joins points (0,1) and (1,1).

This implies that C is not path connected. On the other hand, it is clear that C\Y is path connected. Why is that? Because if you take any point in C\Y. We have deleted Y, so any two points will look like this. So, two points over here, so we can first come from here to the x-axis, and then we can travel to this point, and then we can go up.

So, we cannot do this if our point is on Y, because the origin has been left out. So, we cannot pass through the origin, it is clear that C\Y is path connected. So, this implies that C\Y is connected and so this implies that the closure of C\Y in C is also connected, because if you take any subspace A, then its closure is also connected. But the closure of C\Y in C is exactly is all of C, because if you take any point on Y, we can find a sequence with the same y-coordinate, and this sequence converges to y, which converges to this point q. This shows that C has two path components, namely Y and C\Y and just one connected component, C.

And this also shows that since C\Y is not closed in C, this implies path components need not be closed. We will end here. Thank you.