Point Set Topology
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Week 01
Lecture 02

Okay, so we shall continue from where we left off in the previous lecture. So, in the previous lecture we saw how to give different topologies on the same set X, and the set X could be any set. But today in this lecture we will see some examples which are going to be more important in this course. So the next example we are going to see is the standard topology on set of real numbers, so the real line. So, recall that by the interval (a, b) $\subset \mathbb{R}$, (a subset of R), we mean the subset of real numbers, those real numbers "x" such that x is strictly (greater) than a, and strictly less than b. So, here the assumption is that a is strictly less than b, and we allow a and b to be $-\infty$ and $+\infty$.

So, if a $=-\infty$, then this is those $x \in \mathbb{R}$, such that $x<b$. Similarly, if b $=+\infty$, then this is those $x \in \mathbb{R}$, such that $x>a$. In particular, $\mathbb{R}$ can be written as $(-\infty,+\infty)$. So we want to define a topology on the real line, so first let us define a property which we will denote by (*).

So let $U \subset \mathbb{R}$ be a subset. We shall say that U satisfies this property (*), so let us see what (*) is: (*): if for every point $x \in U$ there is an $\varepsilon>0$, so I just want to emphasize that the ε depends on x, such that when we take the interval $(x-\varepsilon,x+\varepsilon)$, this is contained in U. So let us see some two easy examples. For example, the interval (0,1) satisfies property (*). Why is this? we take any x which lies in this interval, so x lies between, so we can always find a small neighborhood.

So, this distance is x and this distance is is 1-x. If we take $\varepsilon = \min\{x/2, (1-x)/2\}$ so for us it will be something like this, then it can be checked easily that the interval $(x-\varepsilon,x+\varepsilon)$ is contained in (0,1). Let us see one more example. However, if we take this interval [0,1), so this is those $x \in \mathbb{R}$ such that x is greater than equal to 0 and strictly less than 1. If we take this interval, it does not satisfy property (*).

Why is that? If we take, the point 0 in this set, there is no $\varepsilon>0$ such that $(0-\varepsilon,0+\varepsilon)$ (this interval is simply equal to $(-\varepsilon,\varepsilon)$) is contained in [0,1). So, there is no $\varepsilon>0$ for which $(-\varepsilon,\varepsilon)$ is contained in this half open interval [0,1), so therefore this interval does not satisfy the property (*). So that sort of gives us an idea of what kind of sets satisfy this property, so we define a topology on $\mathbb{R}$ by letting $\mathcal{T}$, so $\mathcal{T} \subset P(X)$ so $\mathcal{T} = \{ U \subset \mathbb{R}$ such that U satisfies property (*)$\}$. So we need to check that $\mathcal{T}$ defines topology on $\mathbb{R}$. We need to check that it satisfies the three defining conditions for a topology.

So let us check these one by one. So recall that, the first condition was that Φ, the empty set, and the entire set should be in $\mathcal{T}$. Clearly Φ is in $\mathcal{T}$ because there is nothing to check there is no x in Φ and therefore there is no condition to check, and so this is vacuously true and it is also clear that ℝ is also in $\mathcal{T}$ as for any $x \in \mathbb{R}$ we can simply take $\varepsilon = 1$ and clearly $(x-1,x+1) \subset \mathbb{R}$. So therefore this first condition, first defining condition for being a topology is satisfied, so for the second condition, we need that if $U_1, U_2, .$

$.., U_n \in \mathcal{T}$ are finitely many subsets of ℝ, then the intersection $\cap U_i$ should also be in $\mathcal{T}$. So let us check that this condition is satisfied, so we need to check that $\cap U_i$ satisfies property (*). $\cap U_i$ should also be in $\mathcal{T}$. So let us check that this condition is satisfied, so we need to check that this intersection $\cap U_i$ satisfies property (*). So let us choose some x in the intersection, in particular so this implies that x belongs to U_i for all $i = 1, 2, .$

$.., n$ and since each U_i satisfies property (*) there exists some $\varepsilon_i > 0$ such that the interval $(x-\varepsilon_i,x+\varepsilon_i)$ is contained in U_i. So then, now let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ So clearly $\varepsilon>0$ because we have finitely many positive real numbers and we take the smallest one among them, so that is also going to be positive.

It is also clear that $(x-\varepsilon,x+\varepsilon)$ this interval is contained in $(x-\varepsilon_i,x+\varepsilon_i)$ because, this is x and let us say this is $x-\varepsilon_i$ and this is $x+\varepsilon_i$ So ε is the smallest one among all these ε_i's, therefore this will be $x+\varepsilon$ and this will be $x-\varepsilon$. So this implies that $(x-\varepsilon,x+\varepsilon)$ is a subset of $(x-\varepsilon_i,x+\varepsilon_i)$ which is contained in U_i and this happens for all i. So in particular this implies that this interval $(x-\varepsilon,x+\varepsilon)$ is a subset of intersection of the U_i's This shows that this intersection of U_i's satisfies property (*). So therefore the second condition for being a topology is also satisfied and finally we have to check one more condition. So let I be a set and suppose for each $i \in I$ we are given $U_i \in \mathcal{T}$.

So then we need to show that the union $\cup U_i, (i \in I)$ is in $\mathcal{T}$, that is, it satisfies property (*). So once again this is easy. We apply the same method that we used to show the second case. So we just take any x in this union. Then there is some $j \in I$ such that $x \in U_j$ Now since U_j satisfies (*) there is an $\varepsilon > 0$ such that $(x-\varepsilon,x+\varepsilon)$ is a subset of U_j.

In particular, this implies that $(x-\varepsilon,x+\varepsilon)$ is contained in U_j, which in turn is contained in the union of all these U_i's. Thus we have proved that the union of U_i's satisfies property (*); that is this union is also in $\mathcal{T}$. So therefore $\mathcal{T}$ also satisfies the third condition. So all these imply that $\mathcal{T}$ defines a topology on ℝ, which we call the standard topology. So in the same way that we define this topology on ℝ we can define a topology on $\mathbb{R}^2$ and more generally $\mathbb{R}^n$.

So let us do the example for ℝ^2 also. The example for ℝ^n will be left as an exercise. So this is our fifth example: The standard topology on ℝ^2 So this example is similar to the previous one. But before that first just the way we have intervals (in ℝ) we are going to define an "open square" of side length 2ε around point (a, b) in ℝ^2 as follows: we define, we will call this S_ε(a,b), this is defined to be those points in ℝ^2 such that | x - a | < ε and | y - b | < ε. So in terms of a diagram this looks like the following: maybe I can make it on the next page.

Suppose we take the point (a,b) here and So, this distance is ε. The center is the point (a,b), this length is ε, and this length is ε. So, this is the picture of this open square and now that we have defined this, let us define the analog of property (*). So we say that a subset U in ℝ^2 satisfies property (*2) Maybe if you want, you can denote it by (*2) so that you do not confuse it with the same (*) earlier, but I will be lazy and I will call this property (*) for every point (a,b) ∈ U there is an ε>0 such that S_ε(a,b) is completely contained inside U. And once again this ε depends on the point (a,b) or it may depend on the point (a,b).

It does not have to be the same for all points in U. So, as before let us take an example. Let us see two examples: of one set which satisfies this property and another which does not. So for example, if we take the set U = { (a,b) ∈ ℝ^2 such that a^2+b^2<1} then we claim that U satisfies property (*).

So, here we have ℝ^2. This is our circle and this has radius 1. Maybe I can indicate the point let us say, (1,0). So, the set U is the region inside the circle, and the point to note is that if you take any point over here which is in this green region, the interior of this circle. Then we can always find an ε>0 such that for any point (a,b) ∈ U we can find an ε>0 such that this set is contained in U. So, this is left as an exercise.

This is the open interval so and similarly similar to the non example we had seen. However, if we take the set V = {(a, b) ∈ ℝ^2 such that a^2+b^2<=1} So, then we claim that V does not satisfy property (*) So why is that? Because in our set V is this, and this is (1,0). Now if you take the point (1,0) then no matter how small we take ε, the set S_ε(1,0) will always go outside this region V. So, if we take the point (1,0) ∈ V then for any ε>0, S_ε(1,0) is not going to be contained in V. So, this is also left as an exercise.

We will stop here, and in the next lecture we will continue with the proof that ***𝒯*** defines a topology on ℝ^2. Thank you.