Point Set Topology

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Week 04

Lecture 19

In the previous lecture, we talked about connected components, and today we are going to introduce the notion of path connectedness, which is a more intuitive notion of connectedness. Let us just see what this is: Definition: (Path connectedness), Let X be a topological space. We say X is path connected, if for any two points x and y in X, there is a continuous map γ: [0,1]->X such that $\gamma(0)=x$, and $\gamma(1)=y$. If you make a picture, our topological space x might be like this, this is x and this is y, and then there is some γ, which could be very complicated, which connects x to y. So, the first thing we want to prove is that We want to prove that a path connected space is connected. Being path connected is a stronger notion than being connected.

Proof: let us assume that X is path connected, but not connected. Then since it is not connected, we can write X as a disjoint union of U and V, where U and V are nonempty open subsets. Since they are nonempty, let x be a point in U and y be a point in V. Then as x is path connected, there is a continuous map γ:[0,1]->X such that $\gamma(0)$ is equal to x and $\gamma(1)$ is equal to y.

Now we simply take the inverse image, so then [0,1] can be written as $\gamma^{-1}(U)$ disjoint union $\gamma^{-1}(V)$. Since U and V are disjoint, the inverse images are also going to be disjoint, and 0 is in $\gamma^{-1}(U)$, because $\gamma(0)$ is equal to x which is in U, and 1 belongs to $\gamma^{-1}(V)$ and both of these are open. Thus we have written [0,1] as a disjoint union of nonempty open subsets. So, they are open because γ is continuous and therefore, the inverse image of an open subset is open. This contradicts the connectedness of [0,1].

So, a path connected space is connected, this completes the proof. Let us see some examples, so [0,1], all these nice examples that we know, are all path connected. I mean if you take [0,1], then for any two points a and b, we can join it by a straight line. So, we can define $\gamma(t) = (1-t)a+tb$. Then γ is a continuous map from [0,1] to [a,b] such that $\gamma(0)=a$ and $\gamma(1)=b$.

So, any two points can be connected by this path. So, the same γ works for showing that $\mathbb{R}$ and $\mathbb{R}^n$ are also path connected. So, for $\mathbb{R}^n$ we can just take two vectors, and we can join them by a straight line. So γ from [0,1] to $\mathbb{R}^n$ and $\gamma(t)$ is equal to a+tb. (a and b are vectors).

So, let us look at S^1, S^1 is this circle over here, so once again given any two points, let us say this is at angle θ_1 and let us take another point here, which is at an angle θ_2. So, this a and this b, then we can connect, the point is we can connect them using a path, which starts at a and goes all the way here. But if you were to write it explicitly, then we can define $\gamma(t)$ is equal to $e^{(1-t)\theta_1+t\theta_2}$. So, we have defined this map, $\gamma(t)$, and we need to show that this is continuous, so we will write it as a composite of two continuous maps. We have this map from $[0,1]$ to $\mathbb{R}$, which is, t goes to $(1-t)\theta_1+t\theta_2$, and then we will take the map from $\mathbb{R}$ to $\mathbb{C}$, which is x goes to e^x.

So, this map is continuous, and therefore, this composite is continuous, so let us try to see that this spheres, so this is S^1, but how about these spheres, so let us just do the example for S^2. For S^2 let us say this is the equator, this is the north pole, let us call it N, and this is south pole, this is S. So, if p is any point which is not equal to the south pole then what we can do is, we can take this point p, we can join it by the straight line. So, first we consider this path $\gamma_1(t)$ that is equal to $tp+(1-t)N$ but now the straight line is not going to lie on the sphere, so $p \neq S$, that is important. We have to make this lie on the sphere, for that, note that 0 does not lie on the straight line, the origin, so then this implies that the norm of $\gamma_1(t)$ is a continuous map from $[0,1]$ to $\mathbb{R}$.

Therefore, we can just define this map $\gamma_1(t)/||\gamma_1(t)||$, this is from $[0,1]$ and the image now will land inside the sphere S^2. What is this map? This map is t goes to $\gamma_1(t)/||\gamma_1(t)||$ this is continuous, $||\gamma_1(t)||$ goes to positive reals, this is continuous because we are taking a continuous map and dividing it by another continuous map. Each of the coordinates of γ_1, so if I were to write this as $((\gamma_1)^1(t)/||(\gamma_1)^1(t)||$, .

..,$(\gamma_1)^n(t)/||(\gamma_1)^n(t)||)$ each of these coordinates is a continuous map because the denominator is never 0. So this shows that N can be connected to any point p using a path. But we want to connect N to S the south pole also, We can just take two paths and thus join them, so we can take this point. First we can connect N to this point and then in the same way that we constructed this path from N to a point p, we can construct a path from the south pole to the point p. So, this gives a combined path from the north pole to the south pole, this shows that, This shows that S^2 is path connected.

How to combine two paths is something which we will see precisely in a minute, and the same proof can be slightly modified. A slight modification of this proof also shows that S^n is path connected for all $n \geq 1$. In fact this proof is better, because then we do not have to worry about x goes to e^x is continuous, we can just do away with that. So, we have seen some examples of path connected spaces, some nice examples where we actually made some pictures and intuitively imagined. But now let us continue with our discussion, So before we continue, we can also ask ourselves, we have seen many spaces so far, What about the path connectedness of those? So what are the other examples we have seen? We have seen $GL_n(\mathbb{R})$, we have seen $M_n(\mathbb{R})$, we have seen the orthogonal groups, special orthogonal groups, we have seen the unitary matrices, SU_n, What about the connectedness of these? So, $GL_n(\mathbb{R})$ is not even connected because we have this determinant map from $GL_n(\mathbb{R})$ to $\mathbb{R}\setminus\{0\}$, which is surjective because we can just take this matrix, determinant of this is -1 and of course, determinant of the identity is 1.

Therefore, $\mathbb{R}\setminus\{0\}$, which is disconnected, we can write it as disjoint union of U and V, where U is this half open interval and V is this half open interval, and this is the point 0, which you are missing out. So the determinant is a surjective map from $GL_n(\mathbb{R})$ to this, in fact we can just put any λ over here, λ nonzero, So we get the surjective map, this implies that determinant inverse of U disjoint union determinant inverse of V, both are nonempty gives a separation, We can write $GL_n(\mathbb{R})$ as the disjoint unit of two non empty open subsets. So, therefore, $GL_n(\mathbb{R})$ is not connected. But on the other hand we can ask, what about $GL_n(\mathbb{R})^+$? This is just equal to determinant inverse of matrices with a positive determinant. We will show that later this is path connected, so $M_n(\mathbb{R})$ is just $\mathbb{R}^{n^2}$ and this is a vector space, since $\mathbb{R}^n$ is path connected, that same proof gives that $M_n(\mathbb{R})$ is path connected.

Once again the orthogonal groups are not path connected for the same reason, because when we take determinant, the determinant lands inside {+1, -1} and this is just {1} disjoint union {-1}, and once again the determinant is surjective, so when we take the inverse image of both these open subsets, That shows that O_n is not connected, so O_n is not connected. It is important to say that the determinant is surjective, because if you look at $GL_n(\mathbb{R})^+$ So let me just write $GL_n(\mathbb{R})$ is not connected. When we look at the determinant map at the level of $GL_n(\mathbb{R})^+$, the image lands inside (-oo,0) disjoint union (0,+oo), but when we take the inverse image of the determinant, determinant inverse of (-oo,0) is empty. Therefore this does not show that $GL_n(\mathbb{R})^+$ is disconnected, so what I am trying to emphasize is, it is important to say that this map is surjective or at least it meets the components of more than one component of the target space. But SO_n, so O_n is not path connected, but SO_n is path connected, similarly U_n and SU_n are path connected.

These three examples are very interesting, but they are also difficult to prove, and to prove these we need the notion of quotient topology, which we will introduce later on in this course. So, we will towards the end of this course we will prove that these three spaces are path connected. And I should say that one should emphasize this result, because it is very interesting, because SO_n is defined as, it has a very complicated definition: A times A transpose is equal to identity and det(A)=1. With this complicated definition 'a priori' if one just looks at the definition it is not all clear if it is path connected, but we can join any two points in SO_n with a path which is completely inside SO_n. Now let us continue as before: we define an equivalence relation on X, We define the relation as follows: We say that $x \sim y$ if there is a continuous map γ from $[0,1]$ to X such that $\gamma(0)=x$ and $\gamma(1)=y$.

This is a new equivalence relation, and let us check that. Let us check that this defines an equivalence relation. So, we need to check three things, First we need to check that $x \sim x$, this is easy, we just take the constant path. So, in other words γ from $[0,1]$ to X, and $\gamma(t)=x$ for all t, and the constant path is continuous. The second is if $x \sim y$ then $y \sim x$, suppose since $x \sim y$, there is a path γ such that $\gamma(0)=x$ and $\gamma(1)=y$.

Then define $\gamma_1(t)=\gamma(1-t)$, so γ_1 is the composite of $[0,1]$ to $[0,1]$, t goes to 1-t, which is obviously continuous, and then here we have γ. So, γ_1 is a path from y to x, so if this is γ, X, this is Y, so γ is going like this, then γ_1 traces the same path in the opposite direction. So, in particular γ and γ_1 have the same image inside X, they are different maps from $[0,1]$ to X, but they have the same image and the third thing we need to check is if $x \sim y$ and $y \sim z$ then $x \sim z$. So, here we will use a theorem that we learnt sometime back, it is about how to check continuity of a map by restricting it to two close subsets. So since $x \sim y$, we have the path $\gamma(1)$ from $[0,1]$ to X, a continuous map such that $\gamma_1(0)=x$, and $\gamma_1(1)=y$, and since $y \sim z$, we have this path, this is γ_2, $\gamma_2(0)=y$ and $\gamma_2(1)=z$.

So, basically this x, this y and this z, so here we have γ_1 and here we have γ_2. Now we define this map h_1 first, We define a map h_1 from $[0,1/2]$ to X. So, $h_1(t)$ is equal to $\gamma_1(2t)$ and then we define a map h_2 from $[1/2,1]$ to X by $h_2(t)=\gamma(2t-1)$. So, let us check that $h_1(1/2)=h_2(1/2)$. So, $h_1(1/2)$ is $\gamma_1(1)$ which is equal to y and $h_2(1/2)$ is $\gamma_2(0)$, which is also equal to y.

Therefore, they are equal. So, now, What does this mean? This means that on the interval $[0,1]$, this half, let us call this set A and let us call this set B. A is the closed interval $[0,1/2]$ and B is the closed interval $[1/2,1]$. To X we have defined this map, so this map is h_1 on this $[0,1/2]$ and it is h_2 on B. So, this defines a map h. This actually defines a map because both h the only point of intersection.

So, h_1 is a map on A and h_2 is a map on B, and both h_1 and h_2 agree on the intersection, in this case the intersection is just $\{1/2\}$. Therefore, it defines a map of sets h from $[0,1]$ to X. So, the question is h continuous? So now, recall our theorem right, so h is continuous. Since A and B contained in $[0,1]$ are closed subsets, so h from $[0,1]$ to X is continuous iff h restricted to A and H restricted to B are continuous. So, obviously A and B have the subspace topology, but h restricted to A is exactly h_1 and what is h_1?, h_1 is a composite map.

So, first we go from $[0,1/2]$ to $[0,1]$, t goes to 2t and then we apply γ_1 to t and γ_1 is continuous. Therefore, their composite is continuous, and similarly h restricted to B is equal to h_2, so what is h_2? First we go from [1/2,1] to [0,1] by t goes to 2t-1 and then we compose with γ_2. So, γ_2 is continuous, this map is continuous, t goes to 2t-1, because we are seeing that all polynomials in the coordinates are continuous. So, therefore, this composite is continuous, so this implies that h restricted to A and h restricted to B are continuous, this implies that h is continuous. So, moreover h(0) is equal to, so h is defined to be h_1 on [0,1/2], so $h(0)=h_1(0)=\gamma_1(0)=x$.

and $h(1)=h_2(1)=\gamma_2(1)=z$. Thus H is a path from x to z. So this shows that $x \sim z$. And this also shows if I have a topological space X, and if I can join two points x and y with a path and y and z with a path, then I can join x and z with a path, which is precisely what we used over here in the example. So, we can join the north pole to any point p and then, and then p, we can join to this point S, so therefore we can join the north pole to S. This defines an equivalence relation on X, the equivalence classes are called the path components of X.

And similar to the proposition about connected components that we proved last time, we have this proposition about path components: So, we can write X as a disjoint union of X_i's. Now X_i's are path components, every path connected subspace of X is contained in some X_i. In fact, it is contained in a unique X_i. (2) each X_i is path connected. So, both these will prove that X_i are maximal path connected subspaces of X.

Notice that earlier we had a third point, which said that when we looked at the connected components, we had that the connected components are also closed, but this is not true for path components. So, we will see a counterexample in the next lecture. Remark: Unlike, so maybe I should mark this in red, this is an important remark. Unlike the connected components, which were closed in X, the path components need not be closed.

closed. So, we will end this lecture here.