Point Set Topology

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Week 04

Lecture 17

So, let us continue with our discussion on connectedness. So, the main result we are going to prove today is the following theorem. Let X and Y being connected topological spaces. Then the product X×Y (with the product topology) is connected. Let us prove this. Let us assume that X×Y is not connected.

So then there are nonempty open subsets A and B, such that this product is the disjoint union of A and B. In the previous lecture, we saw that the image of a connected topological space under a continuous map is again connected. So, the precise result was f from X to Y, X is connected, then f(X) with the subspace topology from Y is connected. Let us use this result.

So we look at the map from Y, so fix x_0 in X, and consider the map from Y to X×Y, given by y +-> (x_0,y). So, to check that this map is continuous, we just need to check that the projections to both factors are continuous. The projection to the first factor is just the constant map which sends everything to x_0 and therefore it is continuous. The projection to the second factor is the identity map which is continuous. So, this map is continuous and has image equal to the subset x_0×Y.

Therefore applying this previous result, thus x_0×Y with the subspace topology, since Y is connected, is connected. Now we intersect this equation with x_0×Y. So, we get x_0×Y is intersected with A, this is an open subset of x_0×Y in the subspace topology, disjoint union with x_0×Y intersected with B. So, if x_0×Y intersected A is nonempty and x_0×Y intersected B is nonempty. Then we get a contradiction to the connectedness of x_0×Y.

So, therefore one of these has to be empty. This implies that x_0×Y is contained in A or in B. It is completely contained either in A or it is completely contained in B. Similarly this happens for all x_0 in X. So, given any x_0 in X, when you look at x_0×Y, that subset is completely contained either in A or in B.

We similarly argue in the same way for every y_0 in Y, the subset X×y_0 is contained in A or in B. So, now we can get a contradiction as follows. So since A is nonempty, let (x_0,y_0) be a point in A. Suppose a point (x_0,y_0) is here. Then if we fix x_0, this is x_0, this is x_0×Y.

This implies that since (x_0,y_0) is contained in A or in B and it has a point, (x_0,y_0), which is a point in $x_0 \times Y$ is in A, this implies that this entire line $x_0 \times Y$ is completely contained in A. Now we can take any point let us say y_1 over here, and we can argue in the same way. We can look at this line. This is y_1 and this line is $X \times y_1$. This implies for any y in Y, (x_0,y) is contained in A and since $X \times Y$ is completely contained in A or in B, and it has this one point (x_0,y).

Let me just take this point y. So, this point is contained in A, and this entire line is contained either in A or B, but this line contains this point which is in A, which forces $X \times y$ to be contained in A and this happens for every y in Y. This implies that $X \times y$ which is equal to the union of $X \times y$'s is completely contained in A, but this shows that B is empty, which is a contradiction. Thus $X \times Y$ is connected. As a corollary we see that $\mathbb{R}^n$ is connected.

So, let us see some applications of the results that we have seen so far. We just showed this corollary, $\mathbb{R}^n$ is connected, and as an application: There is no surjective continuous map f from [0,1] to let us say the [0,1] disjoint union [3,4]. Why is this? Because if there was such a surjective continuous map, then it will force that the image is connected, If such a map existed, then since it is surjective, this will imply that [0,1] disjoint union [3,4] is connected, but clearly that is not possible, because if we let X equal to this disjoint union and we let this be U and this be V. Then both U and V are open in X. X obviously has, I mean we are giving X the subspace topology from $\mathbb{R}$, has the subspace topology from $\mathbb{R}$.

In particular, these two spaces cannot be homeomorphic. Another simple observation is that if X and Y are homomorphic topological spaces then X is connected if and only if Y is connected right. This is easy because if X and Y are homeomorphic, that means there is a bijective continuous map from X to Y. Therefore the image of f is all of Y, and since X is connected, that means Y is connected, and conversely, if you are given that Y is connected, then we can take the inverse of f, and once again, since the image of g is all of X, it will mean that X is connected. Now the next application we have in mind is to show that the spheres are connected, but for that we need the following lemma.

Let T_1 and T_2 be connected subspaces of a topological space X. Assume that T_1 intersection T_2 is nonempty. Then the union of T_1 union T_2 is nonempty So let us prove this. Let us assume, if possible let T_1 union T_2 not be connected, be disconnected. So, let us say our T_1 is like this and T_2 is like this.

So, that means we can write T_1 union T_2, let us call this space Y. Since we are assuming that Y is disconnected, we can write Y is equal to, there are two open subsets. So, every open subset looks like Y intersection U, where U is an open subset of X, disjoint union Y intersection V where U and V are open subsets of X. Let us take a point in the intersection. So, let A be a point in the intersection, which exists because we are assuming that the intersection is nonempty.

So, then A is either in Y intersection U or in Y intersection V. So, assume that A is in Y intersection U. Now let us look at T_1. So, intersecting this equality with T_1, we get that T_1 intersected U disjoint union T_1 intersected V. Now T_1 is connected, as T_1 is connected, one of these has to be empty, that is T_1 is completely contained in U or T_1 is completely contained in V.

Since T_1 contains A and A is in U this implies that T_1 is completely contained inside U Similarly as A is also in T_2, this implies that, and T_2 is also connected. So, repeating the same argument, this implies that T_2 is also contained, but then this implies that Y which is equal to T_1 union T_2 is also contained in V. So, this implies that this Y intersection V has to be empty because Y intersection U is equal to Y. So, therefore this is forced, has to be empty, this is a contradiction. So, now let us use this lemma to show that the spheres are connected.

So corollary of the lemma: S^n is connected. So, to show this recall the map, a projection from a point and using this map we had shown that, we had proved (this was left as an exercise actually) So, this map is ϕ from $S^{(n-1)}$. Let us make a picture. So, we are taking the sphere, we remove the north pole, and then we project to this plane. We have shown that $\phi: S^n \setminus \{N\} \to \mathbb{R}^n$ is a bijective continuous map, whose inverse is also continuous.

So let ψ denote the inverse. As $\mathbb{R}^n$ is connected, let us call this inverse let us call this ψ_1, as $\mathbb{R}^n$ is connected this implies $\psi_1(\mathbb{R}^n)$, which is equal to this sphere minus the north pole, is connected. Now similar to the projection removing the north pole, we can instead remove the south pole, and project once again. So, let us call this map ϕ_1, and similarly to this map ϕ_1 we have a map ϕ_{-1} from the sphere, we remove the south pole, to $\mathbb{R}^n$. So, geometrically what does this map do? We have this sphere, we are removing this point, given any point on the sphere.

So, this point is p and this point is x, we join p and x with a straight line, and x is sent to the point q. So, this is $\mathbb{R}^n$, this is S^{n-1}. So, in the same way we proved that ϕ_1 is bijective continuous with continuous inverse, we can prove that ϕ_{-1} is bijective continuous, with continuous inverse. Let ψ_{-1} denote the inverse of ϕ_{-1}. Then $\psi_{-1}(\mathbb{R}^n)$ is connected as $\mathbb{R}^n$ is connected.

So, our S^n, we can now write it as $\psi_1(\mathbb{R}^n)$ union $\psi_{-1}(\mathbb{R}^n)$ So, $\psi_1(\mathbb{R}^n)$ is S^n minus the north pole and $\psi_{-1}(\mathbb{R}^n)$ is S^n minus the south pole. As both these intersect. Let us call this T_1, as S^n minus the north pole and S^n minus the south pole have non empty intersection, and these are connected, the previous lemma implies that S^n is connected. Of course always $n>=1$ over here.

So, that is a nice application. So, we will end this lecture here and in the next lecture we will continue with connectedness.