Point Set Topology

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Week 04
Lecture 16

In the previous lecture, we introduced the notion of connectedness, and some of the most basic spaces are the closed intervals [a,b] inside $\mathbb{R}$. Today, we will prove that these are connected. Let us begin. Proposition: the interval [0,1] is connected. Proof: let us assume that [0,1] is not connected. Then there exist nonempty open sets U and V, such that, these are also disjoint, [0,1] is the disjoint union of U and V.

We may assume one of these contains 0. So, we may assume that it is in U, 0 in U. Let us consider the set S, is equal to those x in [0,1], such that this entire interval [0,x] is contained in U. So, if you are to look at a picture.

This is our interval [0,1]. Our U may be some combination of open sets, and V is also some open subset, such that U and V are disjoint. Let's say U is both, it could be a complicated open set, we do not know what it looks like, but it contains 0. So, this S is defined as the set of all those x such that this entire closed interval [0,x] is contained in U. So, clearly S is nonempty, because 0 is in S, because as the closed interval [0,0], this is just equal to {0}, this is definitely in U.

Now, let a be the supremum x. We take the supremum over all elements of S. So, we claim that a is in U. By the definition of supremum, by the definition of supremum, there is a sequence of x_n in U such that a_n converges to a. let us call this a_n.

Now, as U and V are open, as V is open in this interval [0,1] and U is the complement of V, this implies that U is closed in [0,1] and since U is closed and each a_n belongs to U, each a_n belongs to U because a_n is in S, implies [0,a_n] is a subset of U. In particular a_n also belongs to U. Therefore, and a_n converges to a. So, U is closed, a_n belongs to U and a_n converges to a. These together imply that a also belongs to U.

We have proved that a is in U. Now, note that as a_n is in S, this implies this interval [0,a_n] is in U, by the definition, which implies when we take union of all [0,a_n] this is also contained in U. Now, observe that, if we make a picture, this is a point a, and we have the sequence of a_n's, a_1, a_2, they do not have to increase necessarily, but they will converge to a, they are going to get closer and closer to a. Now, if you take any point y which is strictly less than a, then, we can choose an ε-neighborhood around a, which does not contain y and for all n sufficiently large the a_n's will be contained in this ε-neighborhood right. So, therefore, this implies that for any y in 0, any $y>=0$ and strictly less than a, we will have that y belongs to $[0,a_n]$ for n sufficiently large.

This shows that the ε-neighborhood shows that [0,a) this half open interval is contained in U, plus we have proved that as we have already proved that a is in U. So, this implies that the interval [0,a], this closed interval is contained in U. So now we claim that, this a has to be equal to 1. If a is not equal to 1, so, if a is strictly less than 1. Once again, we have 0, we have 1 here, and let us say a is somewhere over here.

Then there is $\varepsilon>0$ such that the interval $(a-\varepsilon,a+\varepsilon)$ intersected with [0,1], this is contained in U. So, we can choose ε such that $a+\varepsilon<1$. We can do this as a is strictly less than 1. So, this implies that this half open interval is contained in U, which implies that [0,a] we already know is contained in U. So, therefore, $[0,a+\varepsilon)$ is contained in U, which implies that $a+\varepsilon<1$.

So, this $[0,a+\varepsilon)$ is contained in U, which implies that. So, this is $\varepsilon/2$, this interval $[0,a+\varepsilon/2]$ is completely contained in U. But this contradicts the fact that, this implies that $a+\varepsilon/2$ is in S, but this contradicts the fact that a is the supremum of elements in S. This implies that $a+\varepsilon/2$ is in S. So, therefore, a is forced to be 1 in particular.

This implies that this entire interval [0,1] is contained in U, which contradicts nonemptiness of V. Thus [0,1] cannot be disconnected. This implies that [0,1] is connected. So, this completes the proof. Let us make a remark.

A slight modification of the above proof. Let us say that the proof shows that when a and b are two real numbers such that $a<b$, then the closed interval [a,b] is connected. This is the proof. We just have to imitate the proof, and that is easy. So, this is left as an exercise, but we can do this in a simpler way, which we will see immediately.

Not immediately, a little later. As a corollary of this let us show that $\mathbb{R}$ with a standard topology is connected. Proof: if not, then we can write $\mathbb{R}$ as a disjoint union of two open sets, where U and V are disjoint nonempty open subsets. We can choose any a in U and b in V, and without loss of generality, may assume that $a<b$. Then intersecting this relation, $\mathbb{R}$ is equal to U disjoint union V with the interval [a,b], this implies [a,b] intersection U disjoint union [a,b] intersection V.

As both sets are nonempty, as a is contained in [a,b] intersection U and b is contained in [a,b] intersection V. So, this gives [a,b], this shows that [a,b] is disconnected, which is a contradiction, because both these are open subsets in the subspace topology on [a,b] and here [a,b] has a subspace topology. So, therefore we have written the interval [a,b] as a disjoint union of two nonempty open subsets. So, which contradicts the fact that [a,b] is disconnected. So, as corollary of this, let us try to understand what are all the connected subspaces of $\mathbb{R}$.

Let Y be a nonempty connected subspace of $\mathbb{R}$. Then Y is one of the following: basically it is an interval. So, a,b are real numbers with $a<b$. Let us just take less than equal to b. [a,b], [a,b), (a,b], (a,b) Either this or [a, oo), (a, oo) (-oo, b) (-oo, b].

Let us prove this. The idea is simple, we just let let a to be we define a to be infimum of y in Y, and b to be supremum of y in Y. Here, we have a real line and our y is some subset. So, we take the infimum of infimum and supremum of all the elements in Y. So, then $a<=b$. Note that a and b are allowed to be +-oo.

If a=b, this clearly implies that Y is just the singleton set {a}. So, I leave this as an exercise, and clearly, in this case Y is connected, and equal to (a,b). If a is strictly less than b then we claim that Y is less than b. So, this the open interval (a,b) is contained in Y. If not, then there exists c which is contained in this open interval, such that c does not belong to Y.

So, a is over here, b is over here and there is some c not in Y, but then this will imply we can write Y as (Y intersection (-oo,c)) disjoint union (Y intersection (c, oo)). Let us check that this is nonempty. So, as a is the infimum of elements in Y, there is a convergent sequence in Y which converges to a, and since $a<c$. There will be elements of the sequence in the interval, elements in the sequence which are strictly less than c. So, this shows that this is non empty and similarly this is nonempty, because b is the supremum of elements in Y.

There will be a sequence of elements converging to b, and since b is strictly greater than c. So, almost all members of the sequence will be greater than c. Thus, this gives a contradiction, the assumption that Y is connected, because we have written Y as a disjoint union of non empty open subsets. So, thus the interval (a,b) is contained in Y. Y has the subspace topology, and in the subspace topology both the sets are open.

And also, notice that Y is contained in [a,b]. So, this is the difference between the two sets, as a is equal to infimum of elements in Y, and b is equal to supremum of elements in Y. Thus we have (a,b) is contained in Y is contained in [a,b]. So, this part is relevant only if a and b are finite, from this, we can easily conclude that Y has to be of the type mentioned.

This is left as an exercise. This result gives us a description of the connected subsets of the real line. Next, we want to see a very useful proposition which talks about the interaction of connectedness and continuous maps. Proposition: let f from X to Y be a continuous map. Assume X is connected. So, then f(X) which has a subspace topology from Y is connected.

The proof is easy. So, let us see. If not then there exist open subsets U and V such that f(X) is the disjoint union of f(X) intersected U and f(X) intersected V. So, these are open subsets in Y. So, let me emphasize that U intersection V need not be empty. What is given? we are assuming that f(X) is disconnected, which means there are two open subsets in f(X), but every open subset of f(X) is of the type f(X) intersection U. So, therefore we can we can only say that f(X) intersected with U and f(X) intersected with V, these two open subsets of f(X) are disjoint.

They need not be empty, this is just a word of caution, and both these are open subsets. f(X) intersection U and f(X) intersected V are nonempty. Note that, this implies that we can easily check that $X=f^{-1}(U)$ disjoint union $f^{-1}(V)$. This is an easy check. Moreover, as f(X) intersected U is nonempty, this implies $f^{-1}(U)$ is nonempty, and similarly $f^{-1}(V)$ is nonempty.

As f is continuous, both $f^{-1}(U)$ and $f^{-1}(V)$ are open. Thus, we have written X as a disjoint union of nonempty open subsets, but this contradicts the connectedness of X. Thus f(X) is connected in the subspace topology. This is the way to do it. So, we will see several useful applications of this proposition that we proved. So, we will end this lecture here.