Point Set Topology

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Week 03

Lecture 12

So, in the previous lecture, we had we had ended the previous lecture with this exercise. So, let us just do this exercise. So, what do we have, let us make a picture of this. So, let us say we have ℝ^n. In our case let us just take ℝ^2 and we fix this point (a, b) and we want to show that the complement of this point, we are leaving out this point, So, we want to show that the complement is an open subset right. So, let us just take any point (x, y) and then, how will you prove this? So, let x_ :=(x_1, x_2, . .., x_m) in ℝ^n be in the complement of {a_} right. What this simply means is that x_ is not equal to a_ which implies that there is some i such that x_i is not equal to a_i.

Let ε be the absolute value of x_i - a_i, and this is positive So, we claim that S_{ε/2}(x_) is completely contained inside ℝ^m \ {(a_1, a_2, ..., a_m)} this So, in other words we are taking this distance ε, this is ε, and we are taking a square, let us just to be safe let us just take ε/4 right. So, we are taking this as ε/4 around this point (x, y).

So, let us prove this. Suppose if y = (y_1 ..., y_m) belongs to S_{ε/4}(x_).

So, then this implies that |x_i - y_i| is strictly less than ε/4 right, but then this implies that y_i it cannot be equal to a_i right because x_i - a_i is equal to ε yeah. Therefore this implies that y belongs to ℝ^m \ {(a_1, a_2, ..., a_m)}.

So, this proves that this thing inside in the complement, given any point in the complement we have found a basic open subset which contains that point and it is completely contained inside the complement. So, therefore the complement is open. This implies that oops I am sorry is a open subset is an open subset. So, this implies that ℝ^m \ {(a_1, a_2, . .., a_m)} is an open subset which implies by the definition of closed subset, the singleton is a closed subset. So, with this let us begin the next lecture. So, in this lecture we will define the closure of a subset Let us first write the definition and then, we will see what this means by means of some example. So, let A contained in X be a subset. So, it can be any subset, does not have to be open or closed.

So, the closure of A is defined as follows. So, A closure is equal to those points x in X, such that for every open set, or rather if U is an open set, U is an open set containing x, then U meets A, $U \cap A$ is non empty. Let us try to understand this by means of an example. So, the simplest examples are on the real line. So, let us take the real line with the standard topology and we can take the interval (0, 1), let us take this interval (0, 1), and let us see what its closure is.

Suppose we take any x which is not 0 or 1, let us say $x < 0$. Then we can find a small neighborhood around x which does not meet this interval A. So, A is equal to this open interval (0, 1) right. Similarly, this implies that; So, this implies that if $x < 0$, then x does not belong to A closure. Because the closure requires that every neighborhood every open subset which contains x should meet our set A right.

And similarly if I take any $x > 1$, then also we can find some ε small enough such that the ε neighborhood does not meet A. So, once again this implies that if $x > 1$ then x does not belong to A closure. And let us make a remark, maybe I should have made a remark here itself, the remark it is completely obvious. It is clear that A is contained in A closure. So, let me just highlight that.

Therefore, in this example A is contained in A closure, and A closure has to be contained in this closed interval [0, 1] that is what we have seen, because if we take anything which is strictly less than 0, then it is not in the closure. If we take anything which is strictly greater than 1, then also it is not in the closure. The only possible points which are which could be in the closure are 0 and 1. Let us see if they satisfy the definition of a closure of being in the closure. If we take 0, then no matter which neighborhood of 0 we take, if we take any open subset.

If U is an open subset which contains 0, then by the definition of the topology, there exists an $\varepsilon > 0$ such that $B_\varepsilon(0)$, the ball of radius ε around 0 is contained in U, but clearly $\varepsilon/2$ belongs to this ball, and $\varepsilon/2$ is contained in (0, 1). This implies that U intersection A is nonempty. Therefore 0 is contained in A closure, and similarly one can check easily that 1 is contained in A closure. Therefore A closure in this case is precisely the closed interval [0, 1]. We can give a slightly more complicated example.

Let us give an example in $\mathbb{R}^2$. We have A. So, this region, A, is equal to those (x, y) in $\mathbb{R}^2$ such that $x^2+y^2 < 1$. and I would not write the details, but you can check that if you take any point (x, y) (a, b) let us say such that $a^2 + b^2 > 1$, then we can find a small neighborhood, which does not meet A. then there exists $\varepsilon > 0$ such that $S_\varepsilon(a, b)$ intersection A is empty.

Similarly we can check that if $a^2 + b^2 = 1$, then for every ε, $S_\varepsilon(a, b)$ intersection A is nonempty right. This will show that A closure is exactly the set (x, y) in $\mathbb{R}^2$ such that $x^2 + y^2 <= 1$. Note that A is obviously contained in A closure. So, A closure is just adding the boundary, this boundary circle to A. As we proceed, we will get more familiarized with this notion.

So, for now let us prove a lemma. So, although we will write a proof, it is good to keep a picture in mind while we prove these statements. So, let A contained in X,like for instance a picture in $\mathbb{R}^2$ will be good enough, be a subset. Then A closure is closed in X.

So, that is why the word. This justifies the name closure. Let us prove this. So, it suffices to show that X \ A closure is open in X. The definition of closed subset was the complement should be open. So, that is what we are going to show that the complement is open.

So, let x be an element in X \ A closure right. What this means is that x does not belong to A closure. Then by definition there exists an open set U containing x, U contains x, and U intersection A is empty. But it follows that, if y is any point in U, then y has an open subset (which is U itself) which does not meet A. Then y has an open subset namely U which does not meet A.

So, thus y is not contained in A closure. Therefore, we have proved that, this implies that U is completely contained inside X \ A closure. Thus for every x in X \ A closure, we have found an open set U_x which contains x, and U_x is contained in X \ A closure. Therefore, this implies that this X \ A closure, We can write it as union of x in X minus A closure of U_x. Each of these is open, and an arbitrary union of open sets is open.

So, this implies that X \ A closure is open. Roughly this says that if you take any point here, We can find this small neighborhood, which does not meet the closure and similarly, If we take any point here, and the complement which is this open region in red is open. Okay so let us prove this proposition. Let us use this lemma to prove the next proposition. A set A is closed if and only if A is equal to A closure.

So, let us prove this. So x is in A bar, if it has the property that given any open subset U which contains x, it should meet A. So, let us prove this. Let us assume first that A is closed. So, we need to show that A is equal to A closure. Since A is contained in A closure, we already know this is obvious, we have to observe this in the remark which followed the definition, it is enough to show that A closure is contained in U.

So, taking complements, we should prove this. To show this, it suffices to show that X \ A closure contains X \ A, and which is what we are going to prove. Okay so let us take x belong to X \ A. As A is closed, it follows from the definition that X \ A is open. So, let us denote this open subset by U.

Then, thus there is an open subset, namely U which contains x, and such that U intersection A is empty. U is defined to be the complement of A. Therefore U intersection A is empty. Thus x does not belong to A closure, A bar if you like, by the definition of this A closure. So, this implies that x belongs to X \ A.

So, thus we have proved this and therefore we have proved this right. This implies that A is equal to A bar, that too is one direction of the proposition. So, for the other, next, let us assume that A is equal to A bar. Then by the previous lemma, so what is the previous lemma say, the previous lemma said that A bar is closed in X, and since A is equal to A bar, So, thus, A is closed in X, which is exactly what we wanted to prove.

We wanted to show that A is closed. So, this completes the proof of the proposition. So, as a corollary, as an easy corollary, let B be a subset of X. Then closure of B closure is equal to B closure. So, if we take closure then taking closure again makes no difference.

So, let us see how to prove this. So, from the lemma, over here, it says that no matter which subset we take, the closure is always closed in X right. So, applying this lemma, we get that B closure is a closed subset, and this proposition, the above proposition, says that B closure is closed implies B closure is equal to B closure closure, implies that B closure is equal to B closure closure. So, applying the previous proposition by taking A is equal to B closure closure (gives us the result). So, we will end this lecture by with two exercises with two easy exercises.

So, let A contained in B be subsets. Then A closure is contained inside B closure. That is the first exercise. and the second exercise is, let Z contained in X be a closed subspace. Then let A be a subset of Z which is contained inside.

So, then A closure is contained in Z. So, let me make a remark. So, exercise 2 and the above lemma imply that A closure is the smallest closed subset containing A. So, we will end here.