An Introduction to Point-Set-Topology (Part II)

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Lecture 06

Compact Hausdorff spaces

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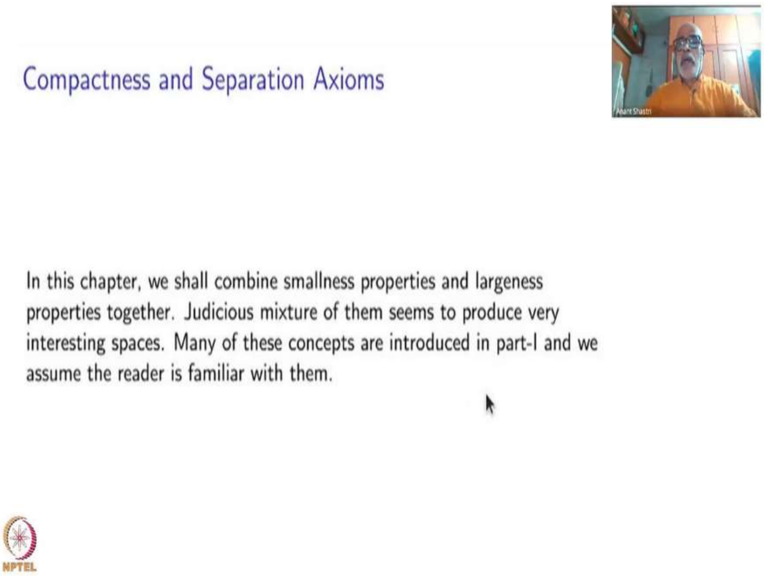

Hello welcome to NPTEL NOC a course on introductory points set topology part 2, module 6. (Please note that module 5 was a live session). We are beginning a new chapter getting right back into point set topology. Recall that in part 1 we have introduced several topological concepts like Compactness, Lindelofness, separability, first countability, and second countability etc, which may all be called some kind of smallness properties.

Then we also have introduced the T_1, Hausdorffness(T_2) regularity, and normality. These are called largeness properties. Now, the idea is to mix them up. Of course, one has to be a bit judicious. A judicious mixture of such things is going to produce many, many interesting results which are also useful.

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So, today we will concentrate upon just one of these things, namely, compactness on one side and Hausdorffness on other side. So, we are going to study Compactness and Hausdorffness together. We have already seen one such application of this one, namely, if you have a continuous bijection from a compact space to a Hausdorff space, then it is a homeomorphism.

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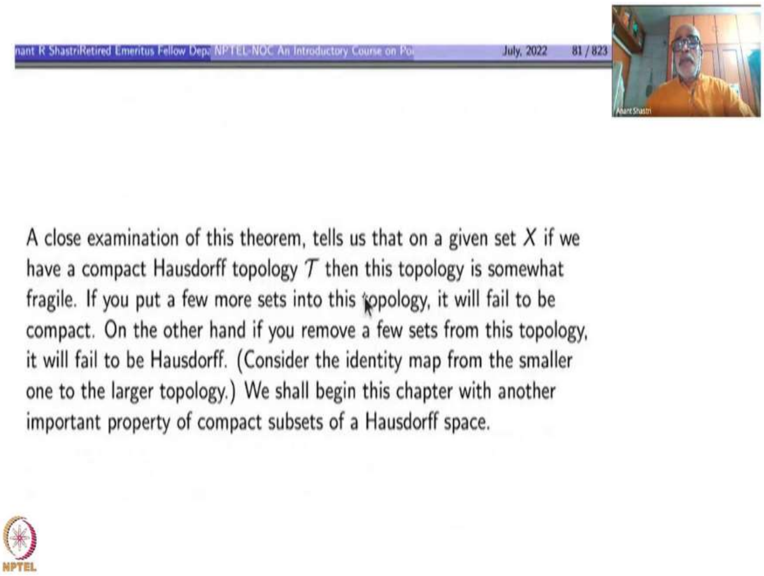

This is what we have already seen in the first part. A close examination of this theorem tells us that instead of having two different spaces, suppose you have the same set on which you have different topologies. So, that then you can compare them by using the inclusion maps from one to the other. If you have a Hausdorff space and then if you take another topology which is larger than that, then we know that the inclusion map will be continuous. From any topology to a larger topology, the inclusion map is continuous.

So, it will be a continuous bijection. So, you can use this theorem to control many things. Namely, if the smaller one is compact, and the larger one is Hausdorff then the inclusion map is a homeomorphism which means that the smaller one is also Hausdorff, and the larger one is compact. Indeed, this just menas that the two topologies are the same.

So, this is the way comparing compactness and Hausdorffness. You know Hausdorffness says there are lots of open sets, the compactness says there are not many open sets in some sense. So, these come together very nicely.

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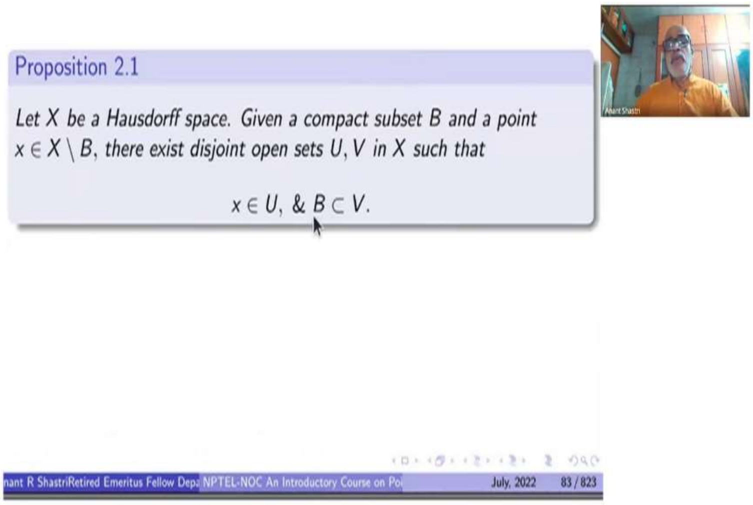

So, here we start with very mild conclusion, start with a Hausdorff space and a subspace which is compact, the same thing as saying that the subspace is both are compact and Hausdorff. So, start with a Hausdorff space X. Given a compact subset B, and a point x not in B, there exists disjoint open sets U and V in X such that x is in U and B is contained inside V.

This may remind you that you have similar property in regularity. So, I am not saying anything new here. Something subset is assumed to be compact not just a closed set here. Then a point x outside this compact subset is taken. They can be separated by open sets, you know disjoint open sets, x is inside U, B inside V. There are people who want regularity and there are those who want Hausdorffness. This will make perfect sense to both. More about this point later-- what is happening here? So, what I am going to do is just go through the proof directly now.

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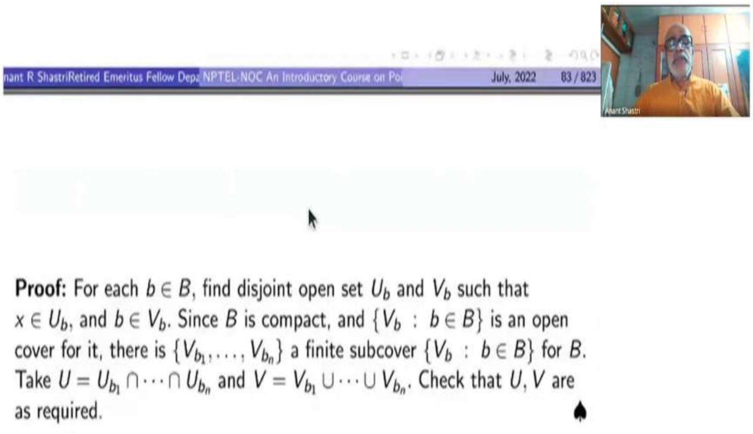

So, start with a point $b \in B$, find U_b and V_b, a pair of disjoint open set such that x is in U_b and b is in V_b. Why this is possible? Because X is Hausdorff that is all. Now, we have done this one for each $b \in B$ and B is compact. These V_b's are open subsets they cover B and B is compact. So, that means that I have some finitely $b_1, b2, \ldots, bn$ so that which V_{b_i}'s cover the set B.

Now, you see for each i, I have a disjoint open set U_{b_i}. So, I take U to be the intersection of these finitely many open sets that is an open subset which will be disjoint from all the V_{b_i}'s and therefore disjoint from their union which I take as V. This union contains the entire of B. Also the intersection contains the point x so, we are done.

So, this method will be repeated again and again. Just watch out this one again. So, what I have done, you have used the compactness of B after using separation by this Hausdorffness we are extracting a finite cover, the finite cover allows us to take intersection of certain other not from not for the cover itself for each member of the other things which are of interest. So, there I take intersection, here I take union. So, we will keep playing this game again also.

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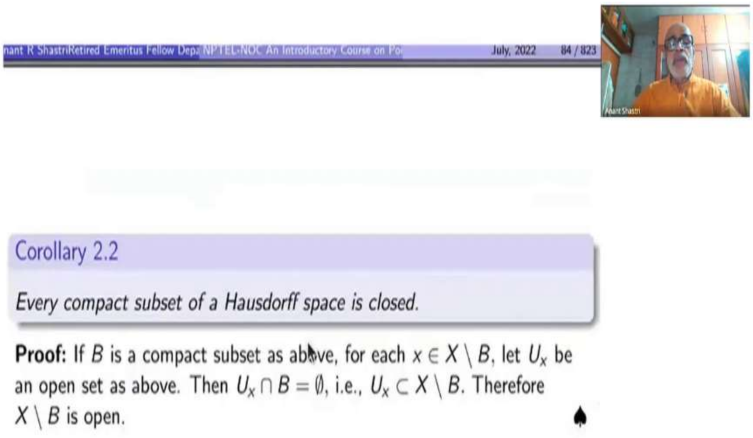

So, as a corollary, every compact subset of a Hausdorff space is closed. Because, for each point x in the complement, I have an open subset U_x disjoint from B. So, if you take union of all these U_x's, as x varies over $X \setminus B$, that will be an open subset which will be precisely equal to $X \setminus B$. That means B is closed. That is the easy corollary.

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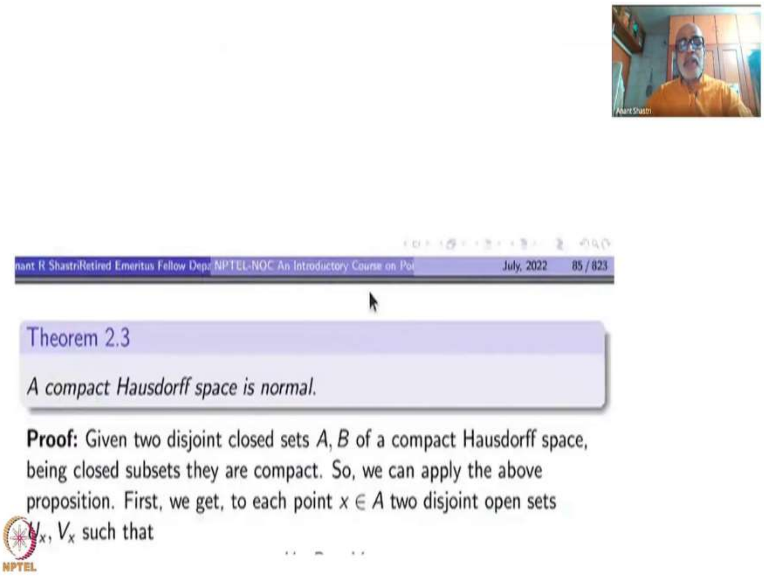

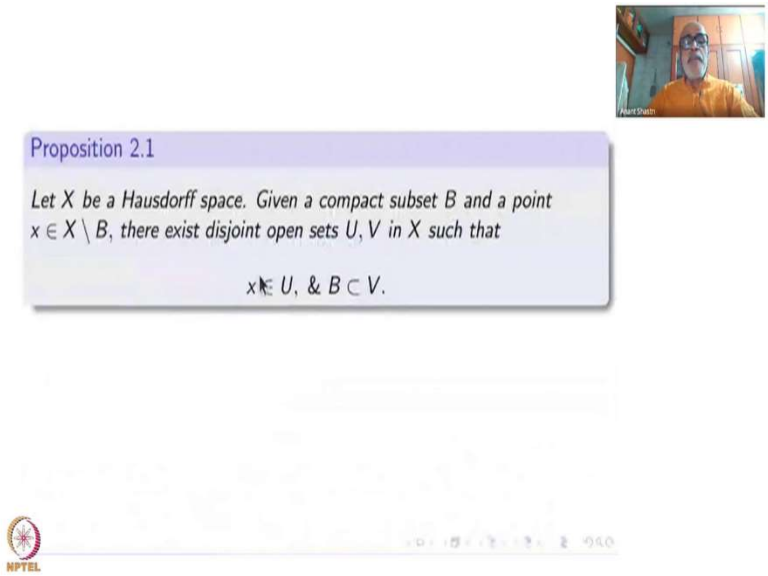

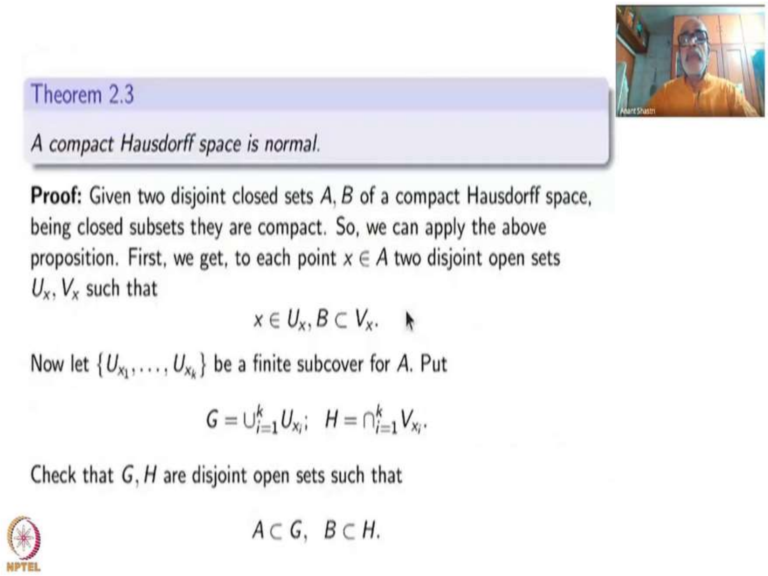

The next thing is what I am more interested in, namely, every compact Hausdorff space is normal. Note that the above theorem, that we have proved actually implies that a compact Hausdorff space is regular. Why? Take any closed set B inside a compact space it will be also compact. Therefore, I can apply this theorem for every point outside B, we will get these pairs of open sets. That means X is regular.

Now, we want to improve on it, namely compact Hausdorff space is normal. Recall that normal means starting with two disjoint closed subsets A and B, I must produce open subsets containing them respectively, say U containing A and V containing B and U and V must be disjoint. So, what we will do? We start applying the proposition.

First for each point $x \in A$, (which is obviously not in B), I get two disjoint open subsets which I will label with, because they depend upon x, U_x and V_x, x is in U_x and the whole of B is inside each V_x. So, I am directly applying the proposition rather than just Hausdorff here. This I can do because, B being a closed subset of X and X is compact so B is compact. Now, let $U_{x_1}, U_{x_2}, \ldots, U_{x_k}$ be finite sub cover of A. How? Because this A is a closed subset of a compact space and hence A is also compact.

So, there will be a finite cover. Again as I indicated earlier, now I take G to be the union of these U_{x_i}'s. That is obviously an open set. But on the other end, I take just H equal to the intersection of V_{x_i}. Since each V_{x_i} contains B, the intersection contains B. Being a finite intersection of open sets, H is open. Since V_x is disjoint from the corresponding U_x, the intersection H will be disjoint from all the U_{x_i}'s. the same kind of argument here. So H will disjoint from G itself. So, what we have proved now that a compact Hausdorff space is normal.

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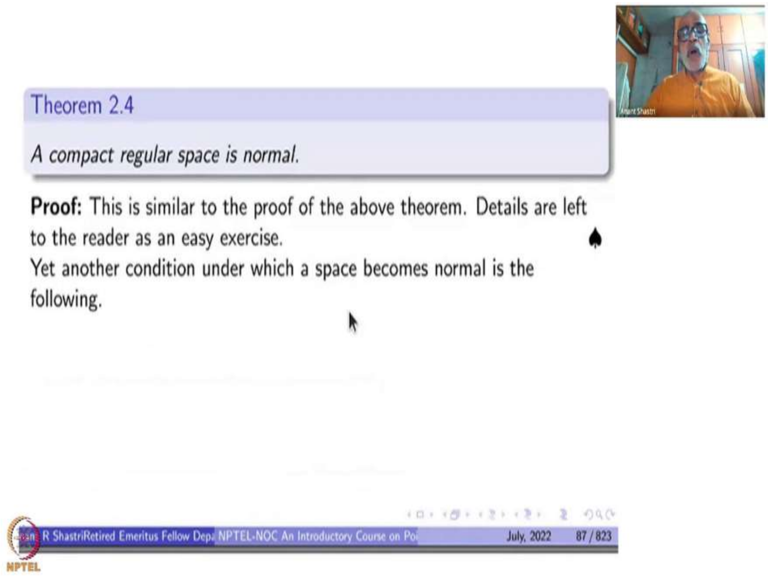

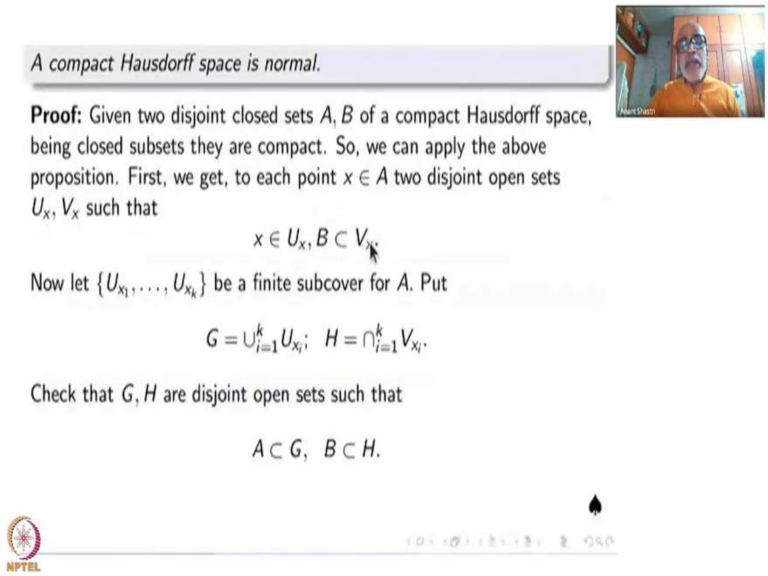

Now, little bit more. Every compact regular space is normal.

Notice that the earlier proposition actually implies that every compact Hausdorff space is regular. That is what we concluded in the proof of the theorem above and later use only this fact. So, I could have just stated that there itself. In other words, this proof of the above theorem actually gives you a proof of the statement that compact regular implies normal. However, you must be careful about this. Compact regular spaces need not be Hausdorff. Now write down a direct proof of this theorem as an exercise.

Yet another condition under which a space becomes normal is the following.

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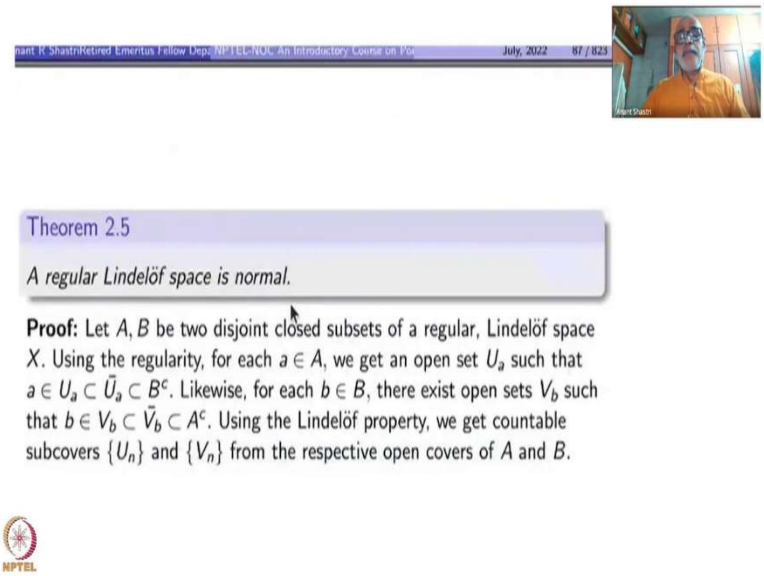

Instead of compactness, you just put Lindelofness which is weaker than compactness. Recall that Lindelofness means every open cover has a countable subcover. That is enough. This is a bit of a surprise because I cannot take intersection of countably many open sets and say that it is open. Yet we can make it work. So, we have sharpened our argument a little bit. How? Let us see.

Start with a regular Lindelof space X and two disjoint close subsets A and B.

Using the regularity for each point $a \in A$, we get an open set U_a, such that a is contained U_a and U_a is contained in $\bar{U}_a$ which is contained in the complement of B. This is because, to begin with A is contained inside the complement of B, which is open and A is closed. This is another version of regularity.

Likewise, for each $b \in B$, there exists an open subset V_b such that b belongs to V_b contained inside $\bar{V}_b$ contained inside complement of A. Here, I am just reversing the roles of A and B, that is all. Now, using Lindelof property, we get countable sub covers $\{U_n\}$ and $\{V_n\}$ for A and B respectively, because A and B are closed subsets of a Lindelof space. (Just like a closed subspace of a compact space is compact, similarly a closed subsets of Lindelof space is Lindelof. This we have seen earlier, it is not difficult. Once I say this one you can verify it easily.)

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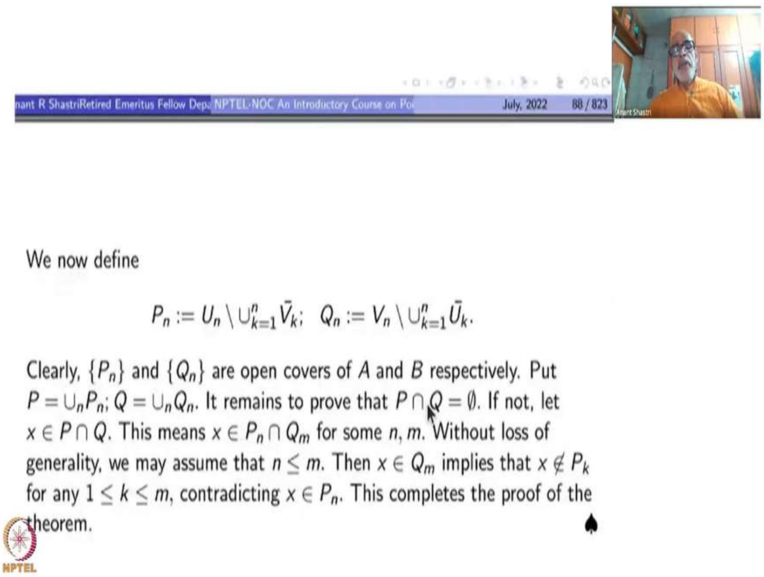
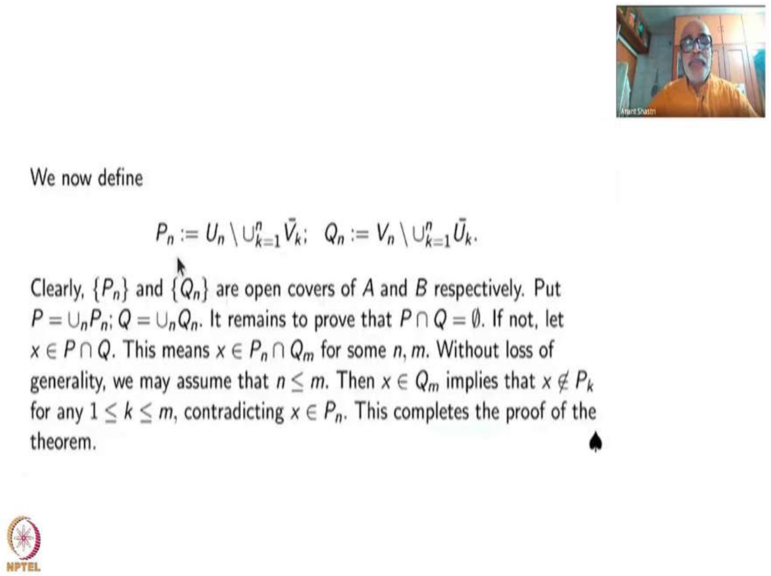




Now, we appeal to a process which is quite common in the study of measure theory. (There are a lot of exchange of techniques between the study of measure theory and the study of topology. I cannot pinpoint whether this particular idea was first used in measure theory or in topology. There are many such ideas.) So, what I do? I start taking P_1 equal to $U_1 \setminus \bar{V_1}$. U_1 is an open set, $\bar{V_1}$ is a closed set. You subtract a closed set from an open set it still open. It is just like the De Morgan law, it is same thing as intersection with the complement here right? taking U_1 minus something.

Inductively, I define P_n equal to $U_n \setminus \cup \bar{V_k}$, k ranging from 1 to n. Union of finitely many closed sets is closed. So, this P_n is open for each n. Similarly on the other side define Q_n is taken as V_n setminus union of k ranges from 1 to n, $\bar{U_k}$.

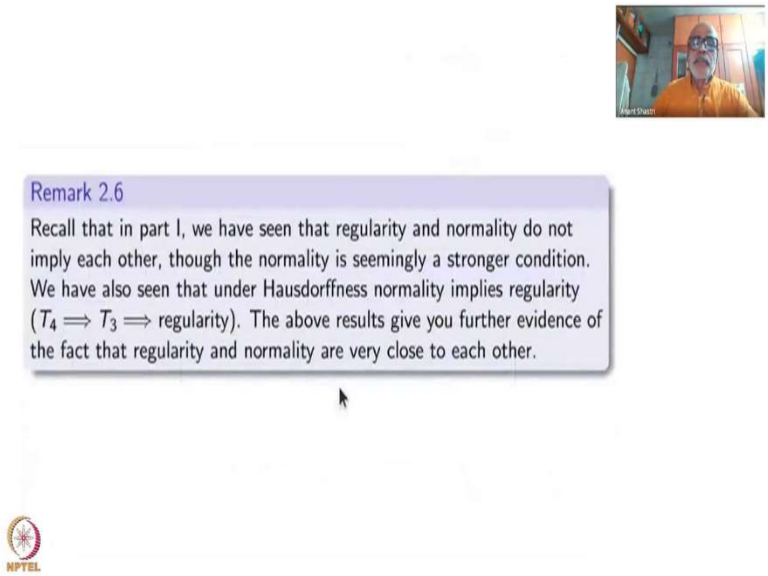
Now, $\{P_n\}$ and $\{Q_n\}$ are open covers for A and B respectively. I have just observed that these are open. Why they cover? Take a point x inside A. It is in one of the U_n's, but I am subtracting closures of some V_k's, here. But each $\bar{V_b}$ is disjoint from A itself being contained in the complement of A.

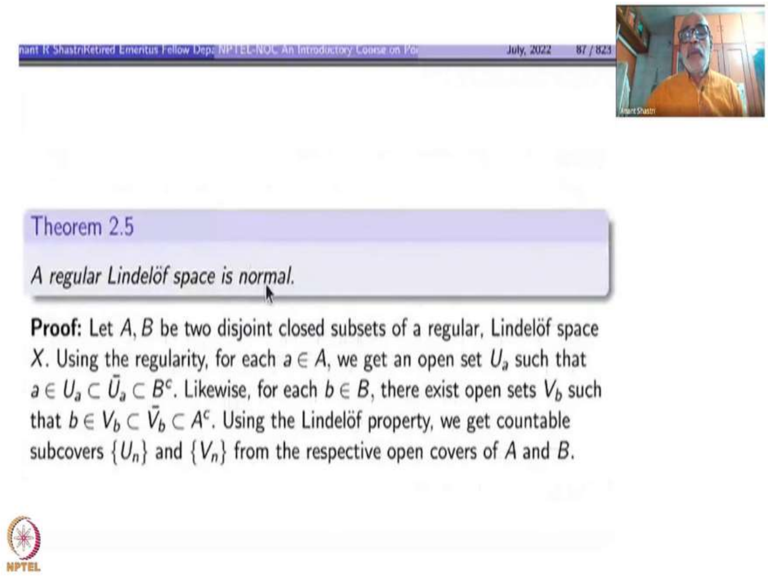
So, when I subtract V_b or $\bar{V_b}$, points of A are not disturbed, they are there. That is why a is in P_n. So {P_n} will cover A. Similarly, $\{Q_n\}$ will cover B. Now take P as union of all these $\{P_n\}$ and Q as union of all $\{Q_n\}$. Then P and Q are open subsets they contain A and B respectively. That is easy.

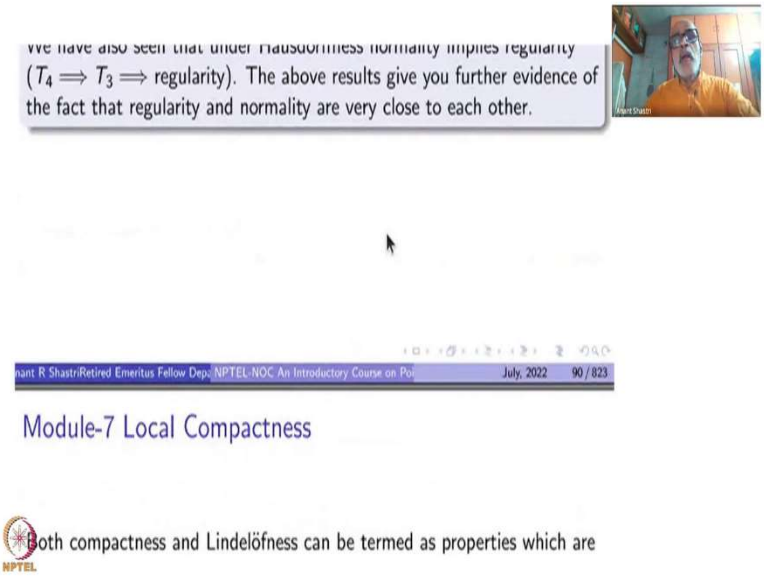
But all this circus you have done precisely to have $P \cap Q$ to be set.

So, let us be convinced with why this intersection is empty. What is the meaning of this is not empty? Take a point x here in both P and Q. This means x must be inside $P_n \cap Q_m$ for n and m, because intersection of unions is union of all these intersections $P_n \cap Q_m$, where both n and m vary. So x must be in one of them for some n and m. Without loss of generality, we may assume that n is smaller than or equal to m. (Otherwise you can interchange n and m.) Now x is inside Q_m implies that x cannot be in any of these $\bar{U}_k$ where k ranges from 1 to m. I can now take $k = n$, then x is not in U_n and hence x cannot be in P_n, which is contradiction. That proves the theorem.

So, this is the modification that we need to handle Lindelofness. Beyond countability, even this trick will fail.(Refer Slide Time: 21:33)







So, here is a remark. In part 1, we have seen that regularity and normality do not imply each other though seemingly normality is stronger because here we are expecting every closed disjoint subsets to be separated in the case of normality, whereas a point and a closed subset are separated in the case of regularity. So, initially, you may think that normality is stronger than regularity. Under some conditions namely T_1 ness, it is true. That also you have seen in part I. (T_4 implies T_3 implies regularity and T_4 is nothing but T_1 plus normality.)

But what I wanted to make it clear is that really, regularity and normality are very close to each other. That is why whenever we are discussing one of them we end up discussing the other one. They are so close by. The results we have proved today is an evidence for this. First we proved that a compact Hausdorff space is regular and then we prove it is normal. So, compact Hausdorff is normal. Once you prove normal of course, it is regular also because it is already Hausdorff. But to prove normality we went through regularity. So, that is the point I wanted to make here, that is all.

So, we will stop here today. Next time, we will bring another new concept, namely local compactness. Thank you.