# An introduction to Point-Set-Topology (Part-2)
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**Week: 12**
**Lecture: 59**
## Classification Of 1-Dimensional Manifolds

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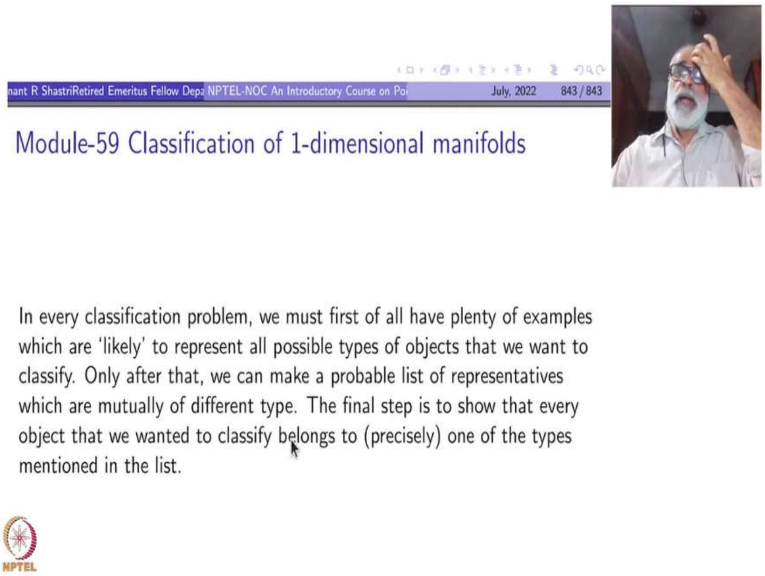


Hello. Welcome to NPTEL, NOC an Introductory Course on Points Set Topology Part II. Today, we have Module 59, a new topic, classification of 1-dimensional manifolds.

In every classification problem, we must first of all have plenty of examples of whatever we are looking for, a complete likely representations of various objects which we want to classify. Suppose, you want to classify a certain number of trees. So, first of all you should have a number of trees, various number of trees and then you can say these are of this type, that is of this type and so on, that is the kind of thing you have to do.

So, which are likely to represent all possible types we do not know yet. So, we think or feel that our list may be just exhaustive, exhaustive of all types. Only after that, we can make a probable list of representatives which are mutually of different type. The final step is to draw a conclusion, is to what, is to show that every object that we wanted to classify belongs to precisely one of the types mentioned in the list. While doing that often what happens is somebody else or you yourself will find out another new object whose type was not listed in your list at all.

So, you have to add that one to the list that is all. This way classification keeps going on. When we were children, biological classification of species was not yet over. By now, several years now, people say it is over now. So, it is like that. Long, long back, Mendeleev started classifying elements. So, he predicted that this is what all the elements will be. So, some elements were not even known to exist. But he predicted them and later they were found. So, it is likely that classification, any scientific classification involves these two steps, fundamental steps here.

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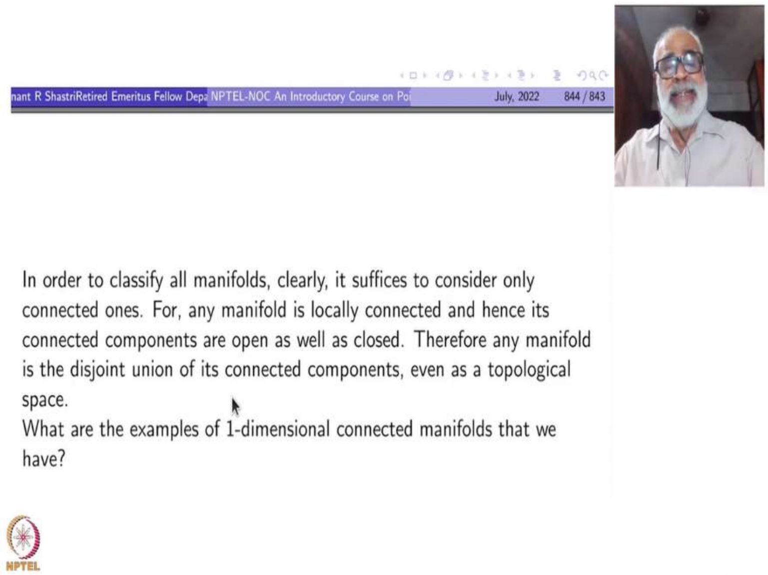


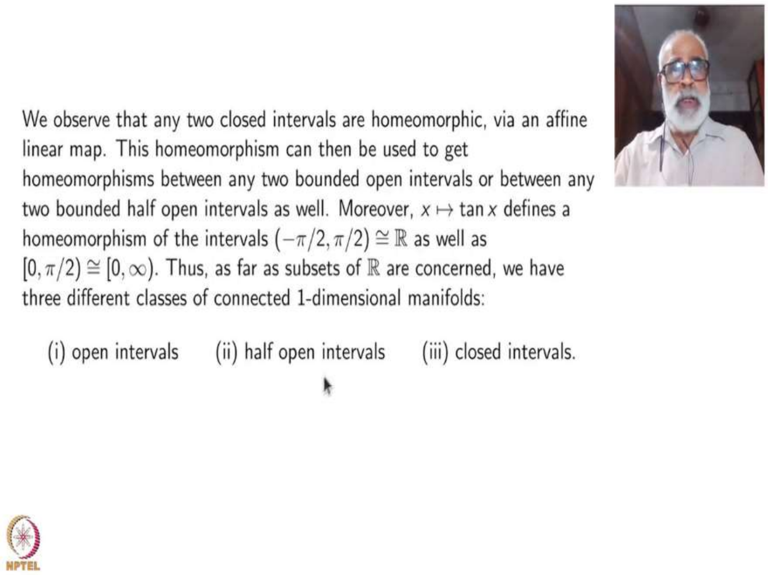


In order to classify all manifolds, clearly, it suffices to consider only connected ones. For any manifold is locally connected, because they are locally Euclidean. And hence, its connected components are both open as well as closed. Therefore, every manifold can be written as a disjoint union of its connected components, even as a topological space. you see when you have disjoint union of topological spaces that is not the same thing as taking a topological space and writing it as a disjoint union of some closed sets. It it written as a disjoint union of open sets, then it will be disjoint union as a topological space also.

So, all this justification is to say that you can only look at connected manifolds. So, right now we are looking at 1-dimensional manifolds.

So, what are the examples of connected 1-dimensional manifolds that you have come across so far? Can you think of more of them? So, this is the first step you have to do.

So, first of all we look at subspaces of $\mathbb{R}$ itself. Observe that any two closed intervals, are homeomorphic to each other other than a being a singleton space. If it is single points single, single points themselves are homeomorphic to each other but they are not 1-manifolds, no problem.

So, via affine linear maps, this homeomorphism can then be used to get homeomorphism between any two bounded open intervals as well, or between any two bounded half open intervals. So, suppose $[0,1]$ to $[a,b)$, I have got a homeomorphism. then I can delete $0$ from $[0,1]$ and its image from $[a,b]$, which is either $a$ or $b$. I will get a homemorphism from open $[0,1]$ close to some half open interval. That is what we have done. And so on.

So, we have we know that all open intervals are homeomorphic to each other, half open intervals themselves will homeomorphic to each other closed intervals are themselves homeomorphic to each other. Further, if you look at $x$ going to $\tan x$, (or some such homeomorphism), this defines a homeomorphism of open interval $(-\pi/2, \pi/2)$ to the whole of $\mathbb{R}$. And if you restrict it to $[0, \pi/2)$, it will give you a homeomorphism onto the closed ray $[0, \infty)$. So, unbounded intervals are also taken care under these three types. intervals, half open intervals, half closed intervals and so on.

So, as far as subsets of $\mathbb{R}$ are considered, we do not have too many connected 1-dimensional manifolds, what are they? open intervals, half open intervals or closed intervals. So, there are only three classes. when I say in the list, whatever I mentioned here, they are themselves not homeomorphic to each other. So, if you look at half closed intervals, $[0,1)$ and $(0,1]$, then there is no need to take both of them in the list, they are already homeomorphic to each other.

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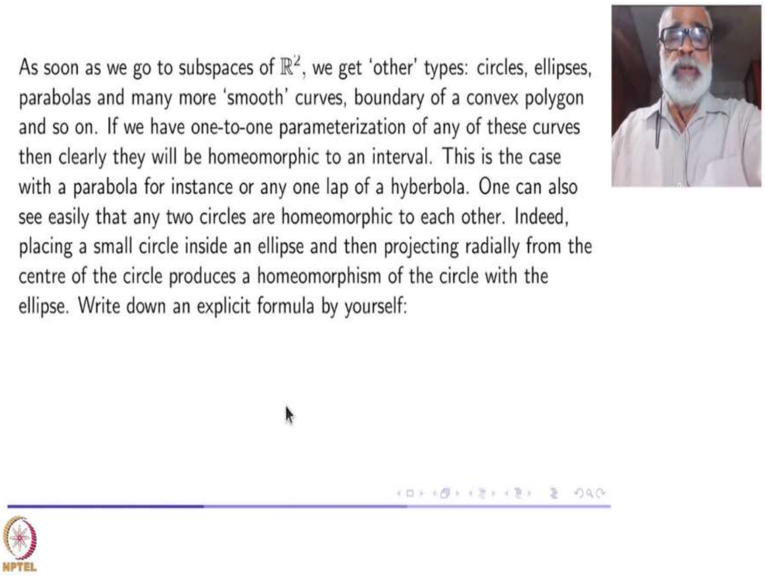


As soon as we go to subspaces of $\mathbb{R}^2$, we get some other types of connected 1-manifolds. It seems like circles, ellipses, parabolas, circles are definitely not there inside $\mathbb{R}$ and ellipses is not there, parabolas are different. There are many more possibilities. Actually there are objects what are called a smooth curves. you can talk about smooth and non smooth this is something completely different game altogether. But simultaneously you can study both of them also.

Look at boundaries of convex polygons, like a triangle or a square or a rectangle or a pentagon. They are also 1-manifolds.

If we can choose a bijective parameterization of any of these curves, that would mean that the curve is homeomorphic to one of the intervals. (First of all, for any smooth curve, each point in it has neighbourhood homeomorphic to an open interval, that is why they are manifolds.) This is the case with a parabola, a parabola can be parameterized completely in a one to one fashion. What is that? In the standard form a parabola is given by the equation $y = x^2$. That means it is the graph of that function itself. So $x \mapsto (x, x^2)$ is the parameterization. But if you look at the hyperbola, hyperbola is not connected. So, you have take only one lap of of the hyperbola then again you can parameterize it, so even if you go to $\mathbb{R}^2$, accept the circles and ellipses, you do not get new objects. However, all circles and ellipses, boundary of any convex polygon etc, are homeomorphic to each other. This is not hard to see. Any two circles themselves are homeomorphic to each other, you try at least this one, write down a formula for a homeomorphism from a triangle to a circle.

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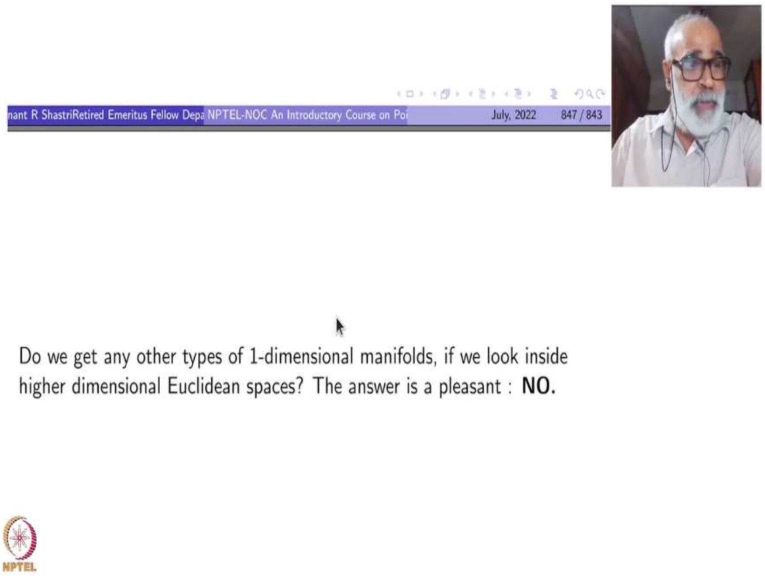


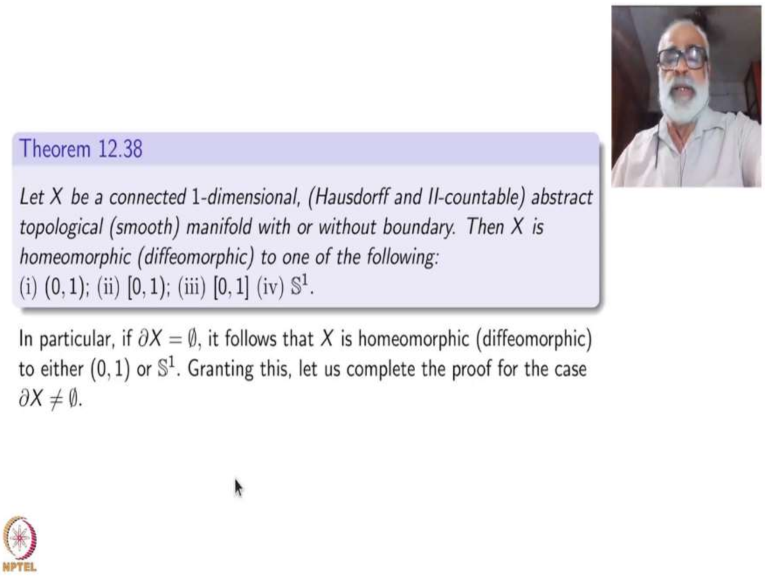


So, do we get any other types of 1-dimensional manifolds if we go to $\mathbb{R}^3$, $\mathbb{R}$4 and so on into other higher dimensional Euclidean spaces? You have to probe. Why I am going inside only Euclidean spaces? because one of our earlier theorems says any manifold is a closed subset of some $\mathbb{R}^{2n+1}$. Therefore, when are hunting for 1-dimensional manifolds you do not have to worry beyond $\mathbb{R}^3$. All 1-dimensional manifolds have copies of them inside $\mathbb{R}^3$.

But in $\mathbb{R}^3$, there may be a weird kind of embeddings of the circle. Do not worry. The very fact that they are embeddings of a circle, means they are only homeomorphic to a circle. $E$ embeddings can be very funny but they are homeomorphic to a circle, we are not bothered about weird embeddings of circles at this moment.

So, are there any other 1-dimensional manifolds? The answer is no. So, that is the gist of whatever is going to come now, maybe today and one more day, tomorrow. So, two more lectures we may have to take. So,  this is the theorem, final theorem:

Let $X$ be any connected 1-dimensional manifold, we just means remember it is Hausdorff and II-countable abstract.

So, topological manifold I have put smooth in the bracket because the statement is true for smooth case also, correspondingly instead of homeomorphism you will have diffeomorphism The final conclusions are the surprisingly they are the same here. But for our purposes we will ignore the smooth path and diffeomorphism path we will be only proving the topological aspect.

So, what are the statement take a connected to 1-dimensional manifolds with or without boundaries specifically I mentioned the boundary case also here then $X$ is homeomorphic to one of the following $1, 2, 3, 4$. What are they open interval, half open interval, closed interval or the circle look at this case.

The first one and the last one are manifolds that means manifolds without boundary the ii one and iii one are manifolds with boundary they are all connected of course otherwise I would not list them here this and this one these are non compact, these two are compact. So, if you want only compact on only these to will get, if you want non-compact ones only these two you will get so do not put compactness you have all the four of them.

Now, first thing what I will do is granting that we know the classification for manifolds without boundary, I will complete the proof of classification for all of them. That means when you allow boundary you will have exactly two more members in the list. that is what I will show you today, granting the classification theorem for the case when boundary is empty.

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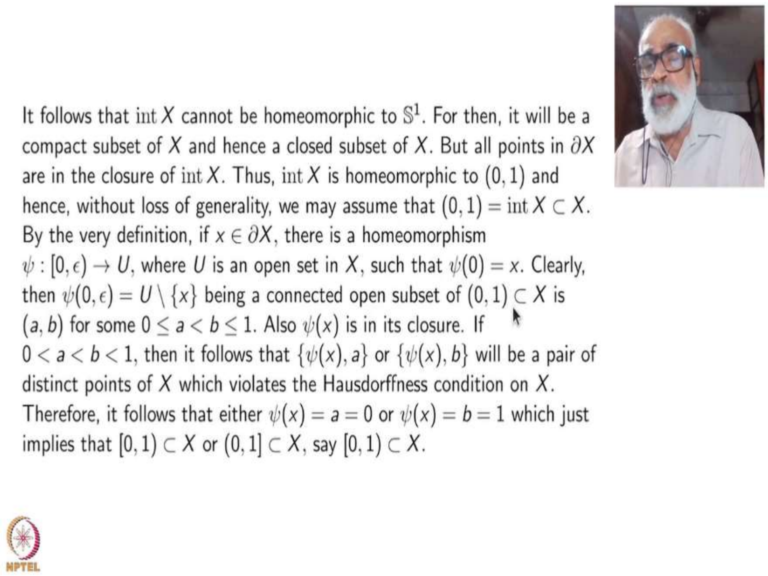


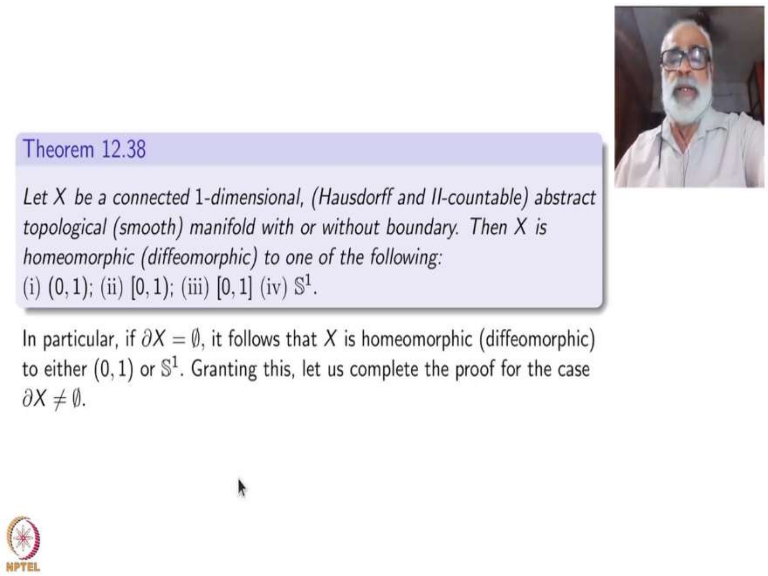


So, start with $X$, a connected manifold with boundary of $X$ non empty. We then know that interior of $X$, being a connected boundaryless manifold, must be either the open interval or the whole circle. But I have assumed that boundary of $X$ is non empty, so, $intX$ cannot be a circle because if it is, then $intX$ is both open and closed in $X$ which means $X$ is disconnected. it is already closed.

So, interior of $X$ must be homeomorphic to an open interval. The question is now, how many boundary points can be there? you put here. I want to say that you can put a boundary point around the two endpoint of this interval, at one of them or both of them and that is it. You put only one of them you will get a carbon copy of a half closed interval, if you put both of them you get a closed interval. So, these are only two distinct cases that is what we have to prove.

In other words, through a homeomorphism, having assumed that $intX$ is actually the open interval $(0, 1)$, I want to show that boundary of $X$ can have at most two points.

By the very definition, if $x$ is a point in the boundary, then there is a homeomorphism $\psi$ from $[0, \epsilon)$ to $U_x$, an open subset in $X$, such that $\psi(0) = x$. This $\psi$ is nothing but the inverse of a local chart at the point $x$.

Now look at $\psi[0, \epsilon)$, the image of the open interval which is contained in $intX = (0, 1)$.

Being a connected open subset of $(0, 1)$, it must be equal to $(a, b)$ for some $0 \leq a \leq b \leq 1$.

Now the point $x = \psi(0)$ is in the closure of $(a, b)$ and not a point of $intX$, the closure is being taken in $X$.

Suppose now that $0 < a$ and $b < 1$. It follows that we have a pair of points $\{\psi(0), a\}$ or $\{\psi(0), b\}$ which violate Hausdorffness. Every neighbourhood of $\psi(0)$ will intersect either every neighbourhood of $a$ or every neighbourhood of $b$.

So, it follows that either $a = 0$ or $b = 1$ or both. In any case, this implies $\psi[0, \epsilon)$ is equal to $[0, t)$ for $t \leq 1$ or $(s, 1]$ for some $s \geq 1$, with $x$ being identified with $0$ or 1 accordingly. Since this is true for every point $x$ of the boundary, it follows that boundary of $X$ has at most two points, if it has one point then $X$ is a half closed interval and if it has two points then $X$ is a closed interval.

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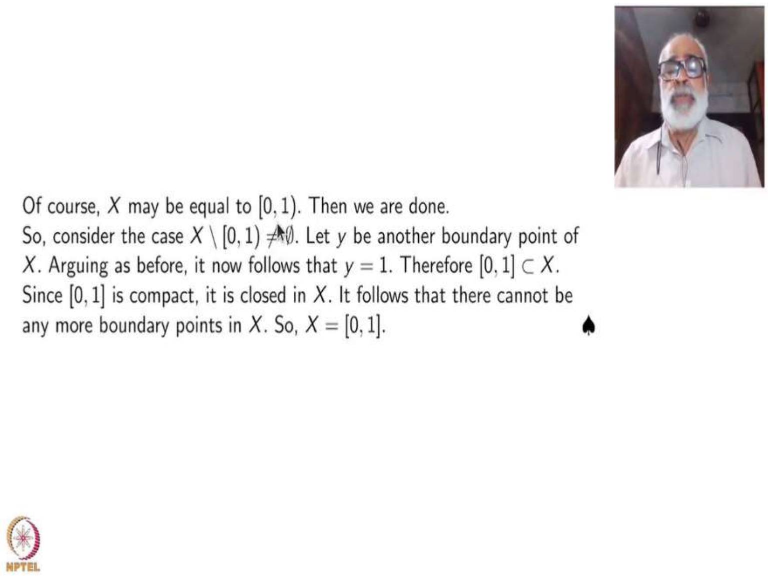

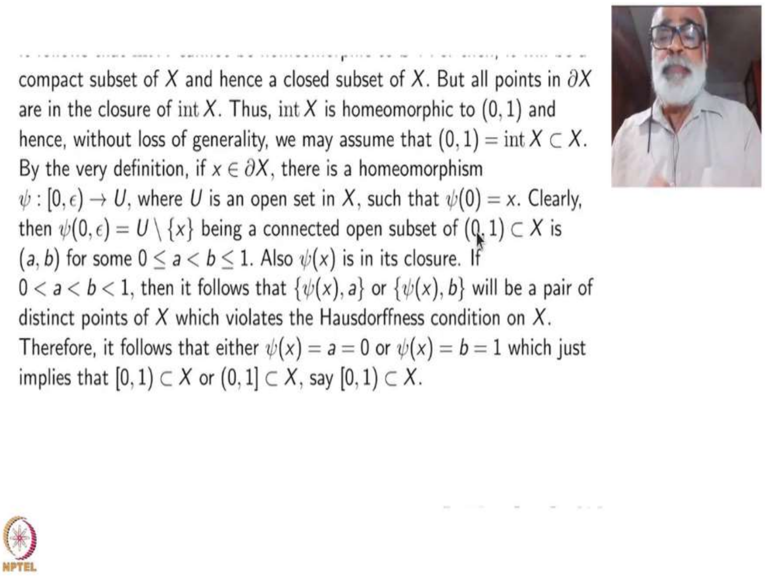


Of course $X$ may be equal to $[0, 1)$. Then the case is over there is no more to bother about this is allowed then we are done. So, consider the case when $X \setminus [0, 1)$ is non empty, there is some more point let $y$ be another boundary point of $X$. Arguing exactly as before, it now implies that $y$ has to be equal to $1$ now, because it cannot be equal to again another $0$ then the $0$ this $y$ will be violating the Hausdorffness, it cannot be in the other end.

**Student:** Sir, I had a question here. So, we had a homeomorphism from open interval $(0, 1)$ to the interior of $X$. So the topology on the interior of $X$ was that of open interval $(0, 1)$. Now when you took a boundary point when we saw that it can be $0$ and so the first line says of course as $X$ can be equal to close intervals $0$ so we have only know that $0$ is the boundary point in the topology on this $[0, 1]$ is also the same which comes from the source.

**Professor:** It has been.. That is precisely what we are doing here now. We are not assuming the rest of the real number system anything to do with $X$. This $(0, 1)$ is a copy of the open interval $(0, 1)$ which is homeomorphic to interior of $X$. It is some space but by our assumption it is homeomorphic to an open interval, this is what we started with. Using the homeomorphism, I have identified $intX$ with $(0, 1)$ a carbon copy of open interval. So, there is the order topology $intX$ now. (But $X$ itself has no order). That order is being used here. So, suppose now the subspace $\psi(0, 1)$ is something $(a, b)$ contained in $(0, 1)$. After removing $\psi(0)$ from $U$, the image of $\psi$. See, see there is a map $\psi$ from $[0, \epsilon)$ to $X$. You do not know where this $0$ is going this $0$ is going where? It is going to $x$ and $x$ is not a point of the interior. It is in the boundary of $X$ and is a closure point of $U \setminus \{x\}$.

If $U \setminus \{x\}$ is as above, there will be a problem with Hausdorffness. You may want to argue that there is a sequence of points in $(a, b)$ converging to $x$, (I do not want to argue with convergence etc here though) which is definitely a point not in the larger interval $(0, 1)$, then you have a problem. That is what you are seeing.

See this is similar to our earlier problem wherein compactification of an open interval $(0, 1)$ was considered. Can you have one point compactification, two point compactification, three point compactification and so on.

The one point compactification of $(0, 1)$, the open interval is the circle. The two point compactification is the closed interval $[0, 1]$. Can you have a 3-point compactification four point compactification, etc.? So, this was a question I think we had left it to you as an exercise. Maybe by now you have solved it. I will tell you the answer now, what is the answer?

If you do not put the Hausdorffness condition on the compactification then it is possible. If you put the Hausdorffness, the same argument as here or you can do it separately without bringing in manifolds at all now, just Hausdorffness, you cannot put a third point at all. Over. Remember a compactification means what? The original space $Y$, must be dense inside the larger space $X$. Just use that property and that $X$ is Hausdorff. You cannot have three extra points. One point is fine two points are fine. So, that is the answer. Just use Hausdorffness, so that is what I have done here. But here I have used even stronger property namely $X$ is a 1-manifold. So it is easier here namely that the whole space I am not assuming that is a compactification I am just assuming it is a 1-dimensional manifold with boundary, since it is a boundary, I already have that namely interior of $X$ closure is the whole of $X$. All the time I am using is that X may not be compact.

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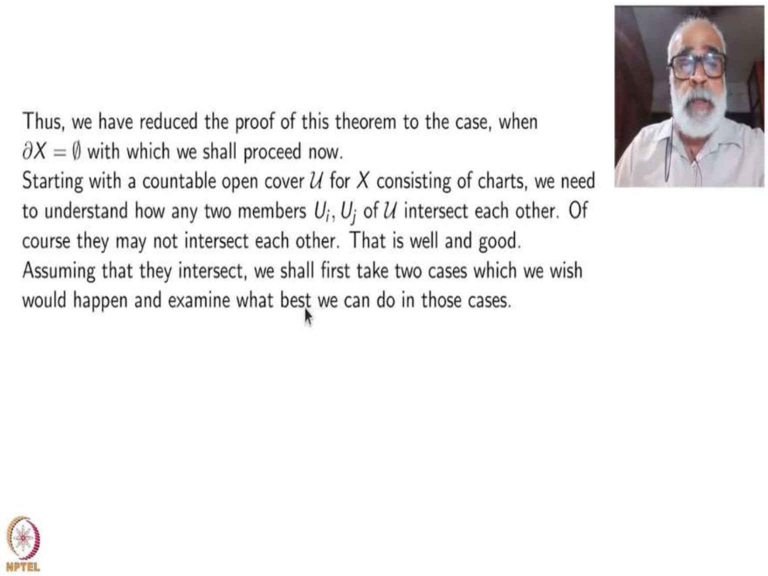


So, what we have done is we have reduced the proof of the original theorem to the case when boundary of $X$ is empty. That means it is a manifold in our original definition with which we shall proceed now. So, let me do a little bit of it.

Starting with a countable open cover $\mathcal{U}$ for $X$ consisting of charts, we need to understand how any two members $U_i, U_j$ of $\mathcal{U}$ intersect each other. Then only you will be able to assemble these various open intervals and produce a new object. How they look like. Of course, they may not intersect each other, that is well and good, no problem. The moment none of them intersects any other each then $X$ will disconnected. So, some of them have to intersect in some way or the other. because whole space $X$ is connected.

So first take just two of them at a time, do not jump to the whole thing yet. We know already that each of them is homeomorphic to open interval and you want to see how they intersect. Note that they are not subspaces of $\mathbb{R}$ now. You see if both are contained in another open interval in $\mathbb{R}$, there is nothing to prove.

Each copy is hanging somewhere. I do not want to use the fact that they are all in some Euclidean space $\mathbb{R}^3$ or $\mathbb{R}^4$ etc. We could use that they are inside $\mathbb{R}^3$, because we have another theorem that the whole $X$ is inside $\mathbb{R}^3$. So, if you want you can draw pictures, no problem. Beyond that you cannot do anything. So, what we have to understand first is how two open subsets $U_i, U_j$, two coordinate neighborhoods, (namely, they are homeomorphic to open intervals) how they intersect in the topological space $X$.

How they intersect? Nobody has told you. So, you have to understand all possible ways, they may intersect. That is precisely the meaning of classification here. So, what we will do is assuming that they intersect, we should first take two cases which we wish to happen and examine what best we can do in those two cases, nice cases. That is to begin with.

But then finally, we should say that no bad cases occur, that is the whole idea. So today we see the nice cases and then later on we shall do the other cases.

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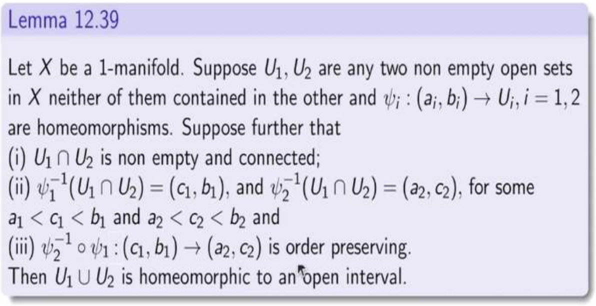


So, start with a 1-dimensional manifold $X$, let $U_1, U_2$ be any two non empty open subsets in $X$ neither of them contained in the other. This is an obvious thing that if one is contained in other there is nothing to bother. You can take the bigger one and you can go ahead to the third one. So, do not get into that kind of cases. Take the case wherein $U_1$ and $U_2$ are both proper subsets of their union, not containing one another, and they intersect. Choose open interval $(a_i, b_i)$ and homomorphisms from $\psi_i$ onto $U_i, i = 1, 2$. Suppose further that

(i) intersection is non empty and connected. (This connectedness is the biggest hypothesis, here. Nonemptyness is anyway has to be there.)

(ii) Now look at $\psi_1^{-1}(U_1 \cap U_2)$. It is some open interval. Why? Because $\psi_1^{-1}$ of an open set is open (and since $\psi$ is a homeomorphism), and connected. A connected open subset of an interval is again an interval say, $(c_1, b_1)$. It is not the whole of $(a_1, b_1)$. Why? Because that would mean $U_1$ is contained inside $U_1 \cap U_2$ is contained inside $U_2$, that should not happen. So, this is a proper open interval of $(a_1, b_1)$. Similarly, $\psi_2^{-1}(U_1 \cap U_2)$, is a proper open interval of $(a_2, b_2)$. So, what I have done, I have actually assumed that the first one is $(c_1, b_1)$. So, the other end is fully taken, $c_1 > a_1$. Similarly, the second one is assumed to be $(a_2, c_2)$. What does that mean? I have taken this end already, the other end $b_2, b_2$ must be strictly bigger than $c_2$.

So, that is what I have written: $a_1 < c_1 < b_1$ and $a_2 < c_2 < b_2$. So, the second assumption is very much stronger. I do not know why it should happen, but I would like this to happen.

(iii) The third condition is not very strong $\psi_2^{-1} \circ \psi_1$ from $(c_1, b_1)$ to $(a_2, c_2)$, ($\psi_1$ takes$(c_1, b_1)$ into $U_1 \cap U_2$, it comes back to $(a_2, c_2)$ via $\psi_2^{-1}$) this must be order preserving. Look at this one. Working inside $X$, I do not have any order. Therefore, I take two homeomorphisms here, take the intersection, from an interval go to the intersection and come back again to an interval. So, this way I am getting a homeomorphism from an interval to interval here. I can talk about whether it is order preserving or order reversing. This is very important, but it can be either of them and both cases can be handled. So, I am assuming it is order preserving, so that the conclusion is very nice, namely, the union $U_1 \cup U_2$ is homeomorphic to an open interval.

So, what has happened is you have an open drawn like this and another like this. So, what you assume there is they are intersecting like this the union will be again an opened interval. So, this portion is there fully this portion is there, it is not something like this open interval that open interval intersecting like that it is not of that nature. Let us go ahead, see whether we can do something. So, we want to prove this union is homeomorphic to an interval.

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**Proof:** Pick up any point $d_1$ such that $c_1 < d_1 < b_1$ and let $d_2$ be the unique point such that $a_2 < d_2 < c_2$ and $\psi_1(d_1) = \psi_2(d_2)$. Put $b_1' = d_1 + b_2 - d_2$. Define $\phi$ by the formula:

$$\phi(t) = \begin{cases} \psi_1(t), & a_1 < t \leq d_1; \\ \psi_2(t + d_2 - d_1), & d_1 \leq t < b_1'. \end{cases}$$

Clearly $\phi$ is continuous and surjective. Because of (iii), it follows that $\phi$ is injective. To check that $\phi^{-1}$ is continuous, observe that on $\psi_1(a_1, d])$, we have $\phi^{-1} = \psi_1^{-1}$. And on $\psi_2([d_2, b_2))$, we have $\phi^{-1} = T \circ \psi_2^{-1}$ where $T$ is the translation $T(x) = x + d_1 - d_2$. ♠

**Lemma 12.39**

Let $X$ be a 1-manifold. Suppose $U_1, U_2$ are any two non empty open sets in $X$ neither of them contained in the other and $\psi_i : (a_i, b_i) \to U_i, i = 1, 2$ are homeomorphisms. Suppose further that
(i) $U_1 \cap U_2$ is non empty and connected;
(ii) $\psi_1^{-1}(U_1 \cap U_2) = (c_1, b_1)$, and $\psi_2^{-1}(U_1 \cap U_2) = (a_2, c_2)$, for some $a_1 < c_1 < b_1$ and $a_2 < c_2 < b_2$ and
(iii) $\psi_2^{-1} \circ \psi_1 : (c_1, b_1) \to (a_2, c_2)$ is order preserving.
Then $U_1 \cup U_2$ is homeomorphic to an open interval.

Very easy proof, pick up any point $d_1$, between $c_1$ and $b_1$. Let $d_2$ be the unique point inside $(a_2, c_2)$ such that $\psi_1(d_1) = \psi(d_2)$. Remember $\psi_1(c_1, b_1)$ is the intersection of $U_1$ and $U_2$ and is also equal to $\psi_2(a_2, c_2)$.

So, if I take a point here, $\psi_1(d_1)$, it will be in the intersection, and so, it is $\psi_2$ of some point where that point must be between $a_2$ and $c_2$. That is all I have done, nothing very special. You pick up one point and choose the other point appropriately, i.e., so that $\psi_2(d_2) = \psi_1(d_1)$. Now you take this number $b_1' = d_1 + b_2 - d_2$.

Now I am going to define a map phi on the interval $(a_1, b_1')$ as follows: On the interal, $(a_1, b_1)$, what I have, I have $\psi_1$ but I do not want to take v on the whole of this interval. I will take $\psi_1(t)$ up to $d_1$. After that I use the $\psi_2(t + d_2 - d_1)$. I can use $\psi_2$ but I have to shift the origin because at $d_1$, these two formulas should agree. When you put $t = d_1$, the parameter for $\psi_2$ should become $d_2$.

So, if you take $s = t + d_2 - d_1$, for $t = d_1$ what happens? $d_1$ and $-d_1$ cancels out and $s$ becomes $d_2$. So they coincide. So, this map is well defined this function $\phi$ is well defined. On each part it is continuous. So, $\phi$ is a continuous function from this interval $(a_1, b_1')$ taking values inside $U_1 \cup U_2$, $\phi(x)$ does not go outside that.

Also, all the points of $U_1 \cup U_2$ are taken care of. If they are inside $U_1$, then $\psi_1$ takes care of up to $d_1$. Beyond $d_1$, $\psi_2[d_1, b_1')$ covers the rest of the points and hence $\phi$ is surjective onto $U_1 \cup U_2$, because of condition (iii). What is it? (iii) says this $\psi_2^{-1} \circ \psi_1$ is order preserving.

(Added by the reviewer: $U_1 \cup U_2$ is clearly equal to $\psi_1(a_1, b_1) \cup \psi_2(a_2, b_2)$. But $\psi_1((d_1, b_1))$ is equal to $\psi_2(d_2, c_2)$ and similarly $\psi_2(a_2, d_2)$ is equal to $\psi_1(c_1, d_1)$, because of $\psi_2^{-1} \circ \psi_1$ is order preserving. Therefore $U_1 \cup U_2$ is equal to $\psi_1(a_1, d_1] \cup \psi_2([d_2, b2)$ which is in turn equal to the image of $\phi$.)

This function $\phi$ becomes injective for the same reason. $\psi_i$ are injective and the images $\psi_1(a_1, d_1]$ and $\psi_2[d_1, b_2)$ have only one point in common viz., $\psi_1(d_1) = \psi_2(d_2)$.

So phi is a continuous bijection.

Now look at $\phi^{-1}$. Observe that on $\psi_1(a_1, d_1]$, we have $\phi^{-1}$ is equal to $\psi_1^{-1}$. On $\psi_2([d_2, b_2))$, $\phi^{-1}$ is not precisely $\psi_2^{-1}$ because there is a shift factor in the parameter, viz, $d_2 - d_1$ is added. Therefore, when you take the inverse, you have to add $d_1 - d_2$, after taking $\psi_2^{-1}$, you translate this one, then what you get the inverse of this map $\phi$. So, both of them are continuous and they agree at the point $d_1$. Therefore, the entire $\phi^{-1}$ is continuous.

So, this proves that the union is homeomorphic to an interval. That was the statement here.

Next time we shall do another thing, but we will wait for that. Another wishful thinking is fulfilled. And after that we will go to the proof of the full classification. Thank you.