An Introduction to Point-Set-Topology (Part II)

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Lecture 56

Manifolds with Boundary

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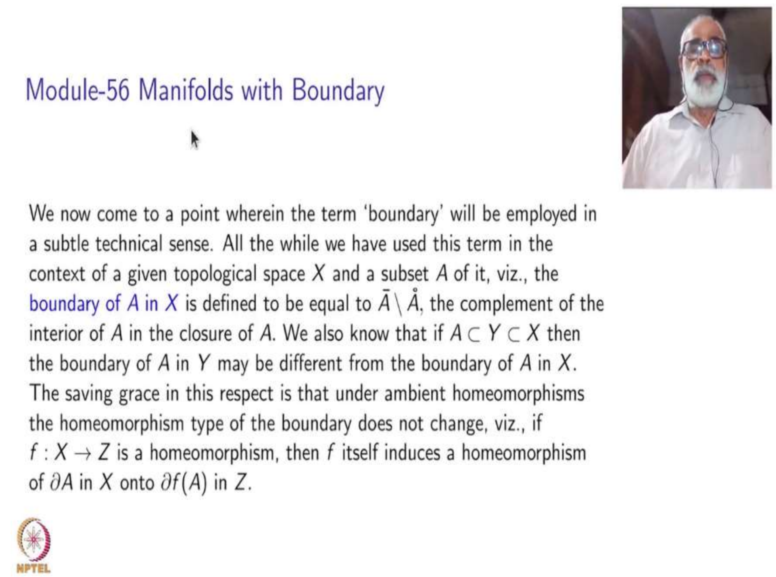

Hello welcome to NPTEL NOC Point Set Topology Part II. Today we will start studying manifolds with boundary. Last time we studied what are called manifolds. So, this manifold with boundary they are also going to be manifolds in some sense. Actually we are going to extend the definition of manifolds to a larger class of topological spaces.

So, I will touch upon the usage of this terminology a little later. The first thing is that we are already familiar with the word `boundary' in a different context. Namely we used it this whenever we have a topological space and a subset of that. Then the boundary of the subset A is defined as the closure of A setminus the interior of A.

This nomenclature for the boundary is totally dependent on the larger topological space X wherein the subset A is sitting. Now, the term boundary will be used in a much subtle way in a technical sense. And it is going to be an invariant of the homeomorphism type of the topological space A, and it does not depend upon where the manifold A is sitting inside.

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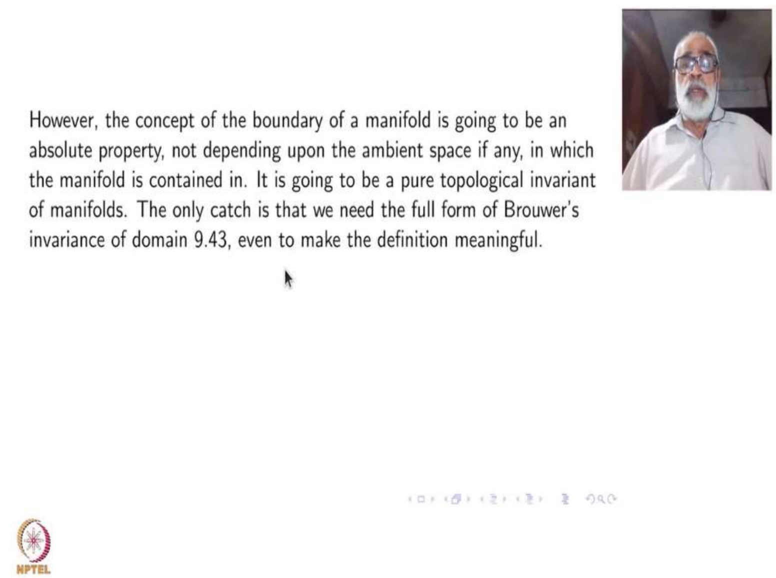

So, having said that much, let us come to the brass-tacks here. The important thing here is even to make this definition we need the full force of Brouwer's invariance of domain.

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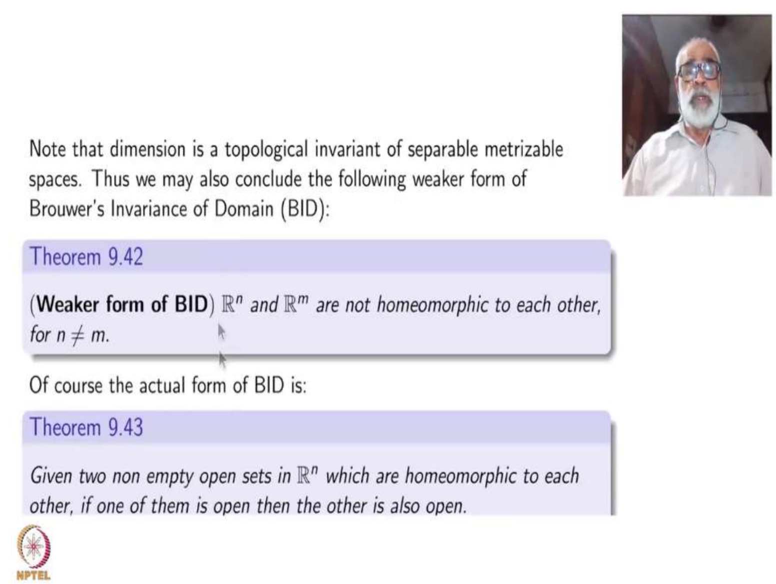

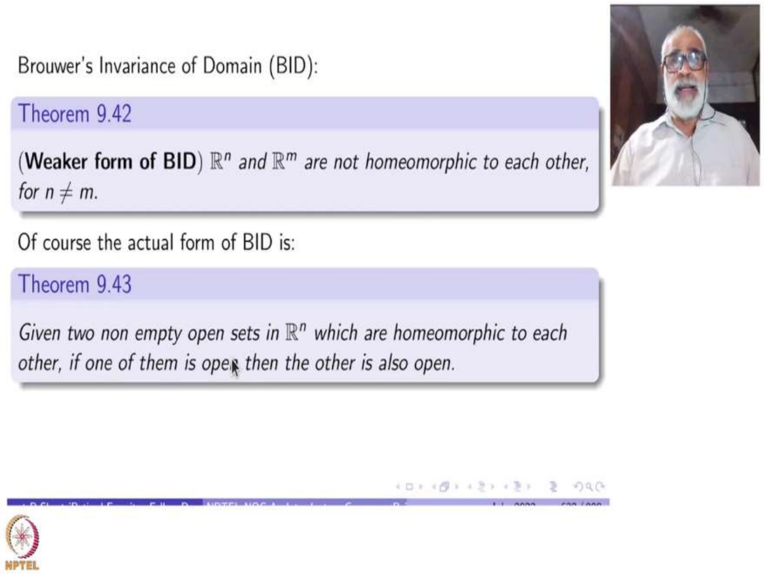

Namely, hope you remember this one, not the weaker form but the stronger for namely, if you have two non empty subsets of $\mathbb{R}^n$, which are abstractly as topological spaces under the subspace topology from $\mathbb{R}^n$, are homoeomorphic to each other. If one of them is open in $\mathbb{R}^n$ then the other one is also open in $\mathbb{R}^n$. This is a completely a non trivial statement, a powerful statement and this is what we will need.

We have used only the weaker version earlier namely, $\mathbb{R}^n$ and $\mathbb{R}^m$ cannot be homoeomorphic if $m \neq n$. That is an easy consequence of this stronger version. So, now, we will need the full force of this. I will come to that again why we need it.

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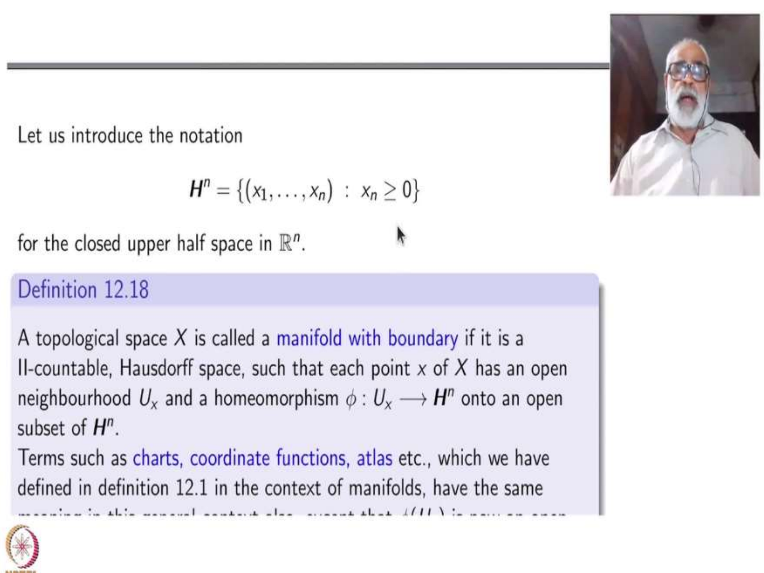

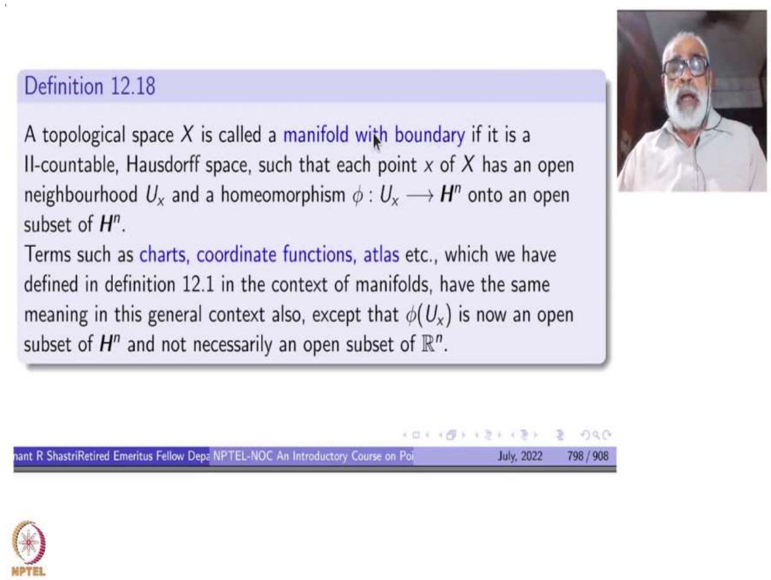

So, first of all our model for manifolds itself will change now. Earlier the model was $\mathbb{R}^n$ or equivalently all open discs inside $\mathbb{R}^n$. Anyway all of them are homoeomorphic to each other. Now, we are taking the half space $\boldsymbol{H}^n$ namely the subspace of $\mathbb{R}^n$ consisting of all points such that the last coordinate of the points is bigger than or equal to 0. You can take any other coordinate also, to get homeomorphic copies. But this one is most convenient one.

So, this is like the ray, when $n = 1$. Closed right ray $[0, \infty)$. So, we are allowing the boundary point $\{0\}$ here the boundary in the older sense namely this point $\{0\}$ will be a boundary point of $[0, \infty)$ inside the larger space $\mathbb{R}$. That is the starting point that is our model now.

So, in $\mathbb{H}^n$ for example, all points with their last coordinate equal to 0, that is a subspace of $\boldsymbol{H}^n$ and is the boundary of this $\boldsymbol{H}^n$ inside $\mathbb{R}^n$. But once we have taken this $\boldsymbol{H}^n$, you can just forget about $\mathbb{R}^n$ and use this model $\boldsymbol{H}^n$ to define our manifolds. So, let us see how.

Let us start with a topological space X. We will call it a manifold with boundary,

(so, now I am not defining a manifold, I am not defining boundary, I am not defining these two words separately here. But I am defining the phrase `manifold with boundary' this entire phrase as a single technical phrase) if X is a II-countable, Hausdorff space and is locally Euclidean in the following modified sense:

For each point $x \in X$, we have a neighbourhood U_x of $x \in X$, and a homeomorphism ϕ from U_x to an open subset of $\boldsymbol{H}^n$.

Of course, the entire $\boldsymbol{H}^n$ is also allowed no problem, as an open subset of $\boldsymbol{H}^n$. Notice that the open set consisting of all point with the nth coordinate strictly positive is homeomorphic to $\mathbb{R}^n$ itself. Therefore, all open subsets of $\mathbb{R}^n$ also have copies inside $\boldsymbol{H}^n$.

In this sense, this modified definition of local Euclidean is more general and allows us little more freedom.

Terms such as charts, coordinate functions, coordinate neighbourhood, atlas etc., which we have defined in the definition 12.1 in the context of manifolds, they all make sense exactly similarly also except that $\phi(U_x)$ may not necessarily homeomrphic to an open subset in $\mathbb{R}^n$, but it is actually an open subset of $\boldsymbol{H}^n$. There is something funny here. In fact, every open subset of $\mathbb{R}^n$ has a copy which is an open subset inside $\boldsymbol{H}^n$ as observed before, $\mathbb{R}^n$ itself is homoeomorphic to the strict open upper half space here, namely take all points such that last coordinate is strictly bigger than 0. So, if you have an open subset of $\boldsymbol{H}^n$ strictly contained inside the interior of $\boldsymbol{H}^n$ in $\mathbb{R}^n$, that is open in $\mathbb{R}^n$ and vice versa any open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of interior of $\boldsymbol{H}^n$.

So, this half space has more open sets than the full space.

Anyway so, we can have these extra opens sub set here of $\boldsymbol{H}^n$, which meet the boundary of $\boldsymbol{H}^n$ in $\mathbb{R}^n$ namely the linear subspace $\mathbb{R}^{n-1} \times \{0\}$. Those are the extra open subsets. Therefore, this definition is clearly a generalization of the old definition of a manifold.

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Denote by $\mathrm{int}\, X$, the set of all those points in x having a neighbourhood U_x homeomorphic to an open subset of

$$\mathrm{int}\, \boldsymbol{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n > 0\}.$$

Clearly this forms an open subset of X and is a topological n-manifold in the old sense. Can you see why $\mathrm{int}\, X$ is non empty if X is non empty? The complement of $\mathrm{int}\, X$ in X is denoted by ∂X and is called the boundary of X. Clearly it is a closed subset of X.

Let us denote by interior of X, the set of all points of X having a neighbourhood U_x which is homoeomorphic to an open subset of interior of $\boldsymbol{H}^n$ itself. You see there are types of points. The first type is our interior of X, consisting of those points for which there is such a coordinate neighbourhood (U_x, ϕ) with $\phi_n(x)$ is bigger than 0.

So, take such points. They will form automatically an n-manifold, namely topological n-manifold in the old sense. So, why the interior is non empty as soon as X is non empty? That is also clear because as soon as x is non empty, there is some open neighbourhood which is homoeomorphic to a non-empty open subset of $\boldsymbol{H}^n$. If you delete the boundary part here that will be an open subset in the interior of $\boldsymbol{H}^n$, which is obviously non empty and so you can take the inverse image of that. So, the same statement you have to see inside $\boldsymbol{H}^n$ first, Then you get it for X.

The complement of interior of X in X is denoted by boundary of X. So, this is just a notation now, I will read it as boundary of X and call it the boundary of X. So, we have used this notation earlier for boundary of A where A is a subset of X. If $A = X$ the entire topological space, then boundary of X is empty.

So, this has no other meaning there. Because the closure is whole of X and interior is also the whole of X. So, boundary of the whole space inside itself is empty.

So, here that is not the case. X is a topological space on its own, it is not contained in anything. This is a topological space on its own. Now boundary of X consists of those points which are not in the interior of X. So, interior of X also has a different meaning here. In the general topological case, interior of X inside X would be whole of X. That may not be the case here. Here also it can happen. But then X will be a pure manifold in the old sense as well. and its boundary is empty.

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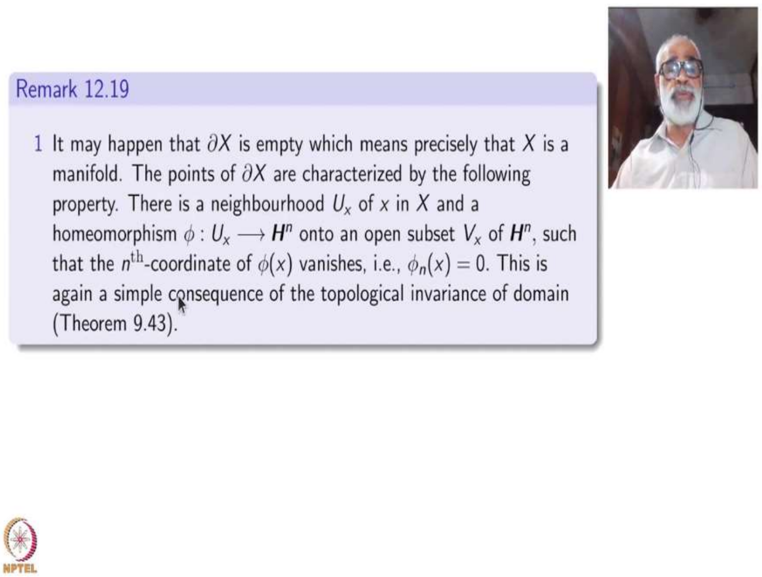

So, I am talking about a special case. It may happen that boundary of X is empty. That means X is a manifold in the old sense. Actually, this is `if and only if'.

The points of boundary of X are characterized by the following property. (So, this is where you have to use the Brouwer's invariance of domain.) Namely, there is a neighbourhood U_x of x in X and a homeomorphism ϕ from U_x onto to the entire $\boldsymbol{H}^n$, equivalently, on to an open subset V_x of $\boldsymbol{H}^n$ such that the n^{th} coordinate of $\phi(x)$ vanishes.

(So, in particular the subspace $\mathbb{R}^{n-1} \times \{0\}$ should be hit by this ϕ.)

So take all possible local charts ϕ for X, and take the union of $\phi^{-1}(\mathbb{R}^{n-1} \times \{0\})$, those are the boundary points of X.

Now, what do you mean by `characterize'? Let me tell you that. So, this is where I have to use Brouwer's invariance of domain.

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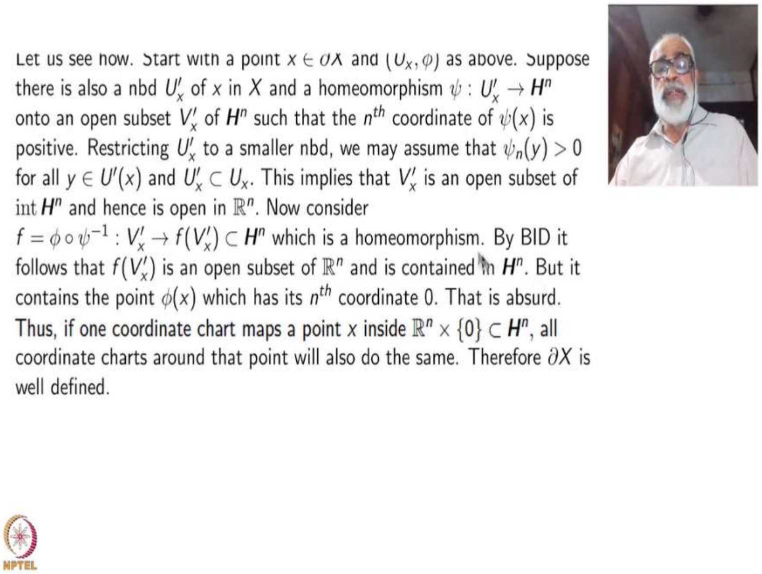

So, let us start at such a point. If possible, suppose there is another neighbourhood U'_x and another homeomorphism ψ from U'_x onto an open subset V'_x of $\boldsymbol{H}^n$ such that the n^{th} coordinate of $\psi(x)$ is positive (either it is positive or it is 0, its greater than or equal to 0 always).

Question is can this happen? The moment one this thing happens, this other thing cannot happen that is the conclusion we are aiming at. That is why this is a characterization.

So, let us see why this cannot happen. Once you have another ψ like this, you can restrict your ψ to a smaller neighbourhood W_x of x, viz., the inverse image $\psi^{-1}(\boldsymbol{H}^n \setminus (\mathbb{R}^{n-1} \times \{0\}))$ which is contained in U'_x.

This will imply that ψ from W_x to $\boldsymbol{H}^n$ is a homeomorphism of W_x onto an open subset G_x of the interior of $\boldsymbol{H}^n$. Any open subset of interior of $\boldsymbol{H}^n$ is open inside $\mathbb{R}^n$ also, because interior of $\boldsymbol{H}^n$ is open in $\mathbb{R}^n$.

Now, consider $f = \phi \circ \psi^{-1}$ from this open subset G_x to $\boldsymbol{H}^n$. Being a composite of two homeomorphisms this f is a homeomorphism of G_x onto a subset say G'_x of $\boldsymbol{H}^n$ which contains the point $\phi(x)$ which has its n^{th} coordinate 0. By Brouwer's invariance of domain G'_x must be open in $\mathbb{R}^n$ itself. So, that is absurd, because no open ball around $\phi(x)$ will be contained in G'_x which is a subset of $\boldsymbol{H}^n$, because such an open ball will have points with their n^{th} coordinate negative. So, the conclusion is that the definition of boundary of X as well as interior of X is now well defined thanks to Brouwer's invariance of domain.

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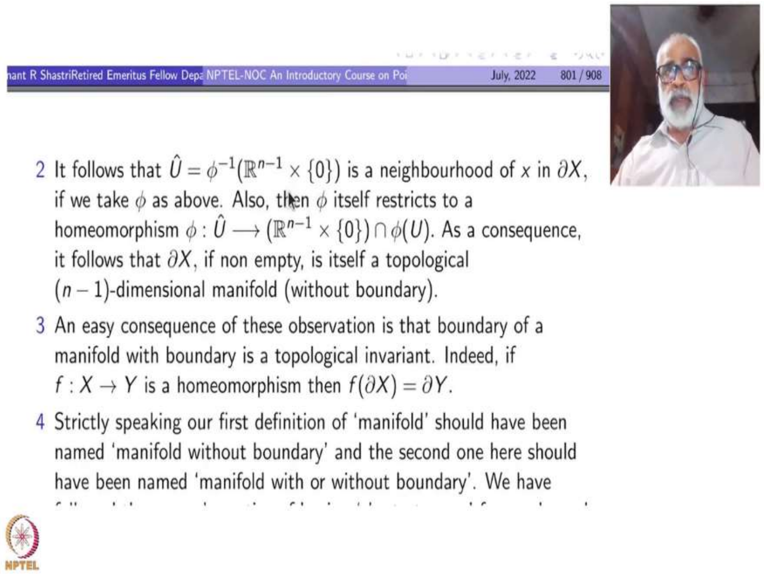

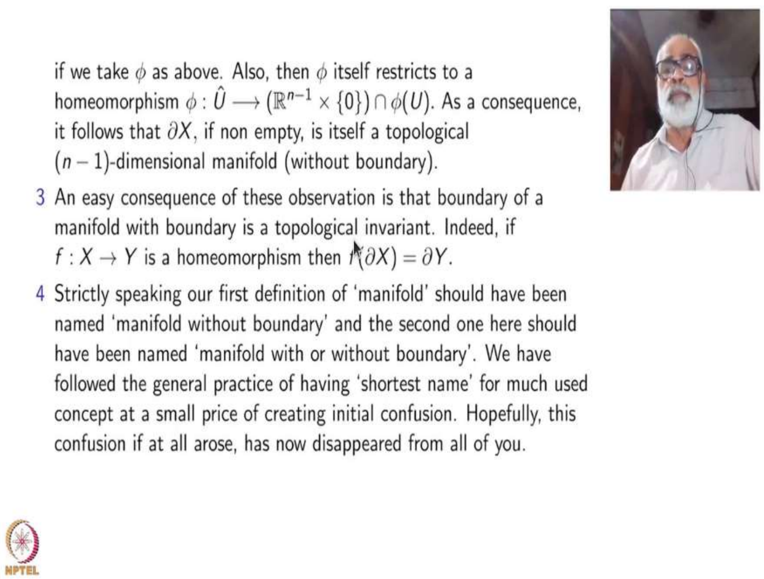

Let us see now. Start with a point $x \in \partial X$ and (U_x, ϕ) as above. Suppose there is also a nbd U'_x of x in X and a homeomorphism $\psi : U'_x \to H^n$ onto an open subset V'_x of H^n such that the n^{th} coordinate of $\psi(x)$ is positive. Restricting U'_x to a smaller nbd, we may assume that $\psi_n(y) > 0$ for all $y \in U'(x)$ and $U'_x \subset U_x$. This implies that V'_x is an open subset of $\mathrm{int}\, H^n$ and hence is open in $\mathbb{R}^n$. Now consider $f = \phi \circ \psi^{-1} : V'_x \to f(V'_x) \subset H^n$ which is a homeomorphism. By BID it follows that $f(V'_x)$ is an open subset of $\mathbb{R}^n$ and is contained in H^n. But it contains the point $\phi(x)$ which has its n^{th} coordinate 0. That is absurd. Thus, if one coordinate chart maps a point x inside $\mathbb{R}^n \times \{0\} \subset H^n$, all coordinate charts around that point will also do the same. Therefore ∂X is well defined.

It follows that $\hat{U}$ which is just a notation for $\phi^{-1}(\mathbb{R}^{n-1} \times \{0\})$, (this may be empty in general, but right now, I am assuming that $\phi_n(x) = 0$, and hence x in there in $\hat{U}$) will be a neighbourhood of $x \in \partial X$, and restricted $\hat{U}$, ϕ itself is homeomorphic to $V_x \cap (\mathbb{R}^{n-1} \times \{0\})$.

So I am taking $\hat{U}$ to be inverse image of $\mathbb{R}^{n-1} \times \{0\}$ and first thing you note is that $\hat{U}$ is contained in boundary of X, because all these points in $\hat{U}$ are coming here in $\mathbb{R}^{n-1} \times \{0\}$. So they are all qualified to be inside boundary of X. So what I am talking? Starting with a (U_x, ϕ) like this, where x has gone to a point inside $\mathbb{R}^{n-1} \times \{0\}$, i.e, $\phi_n(x) = 0$, take $\hat{U}$ equal ϕ inverse image as above, that will be a neighbourhood of x inside boundary of X and homoeomorphic to, via the same ϕ, to an open subset of $\mathbb{R}^{n-1} \times \{0\}$. That mean that boundary of X itself is a manifold, pure manifold in the older definition, of dimension one less namely, of dimension $n - 1$. This is under the assumption that it is not empty. That is all. Boundary of X being empty is also allowed.

And this boundary ∂X itself will not have any boundary.

An easy consequence of all these observations is that the boundary of a manifold is a topological invariant. What is the meaning of that? If you have a homeomorphism f from X to Y, and X is a manifold with boundary then Y will be a manifold with boundary and f of the boundary of X will be equal to boundary of Y. Think about it is not difficult at all. (Of course this also implies that if boundary of X is empty then so is boundary of Y.)

So boundary always is mapped onto boundary of Y by the homeomorphism. So, restricted boundary of X, it is again a homeomorphism.

Now, I come to again this nomenclature about manifolds somewhat apologetically. Strictly speaking, our first definition of manifold should have been named `manifold without boundary' and the second one here should have been named just `manifold' or `manifold with or without boundary'.

The only problem is, right from the beginning before defining the manifold, I have to define what is the meaning of `with boundary' and `without boundary'. So, one does not like that one. The second point is that there is a standard convention of using the smallest word, smallest phrase, to represent the most commonly used concepts. Since we will be studying manifolds and quite often without boundary, so, those should be just called by the shorter name, so we have called them manifolds. So that it is all the explanation for why we are making this one. In any case, if there is an initial confusion about this what is a manifold, what is a manifold with boundary, and what is a manifold without boundary, having spent sufficiently enough time now and got explanation, I hope this confusion if it all has disappeared now.

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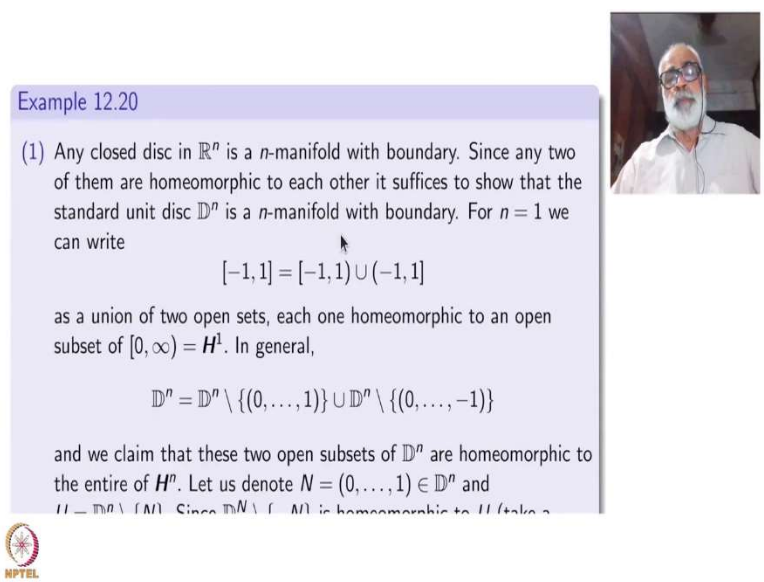

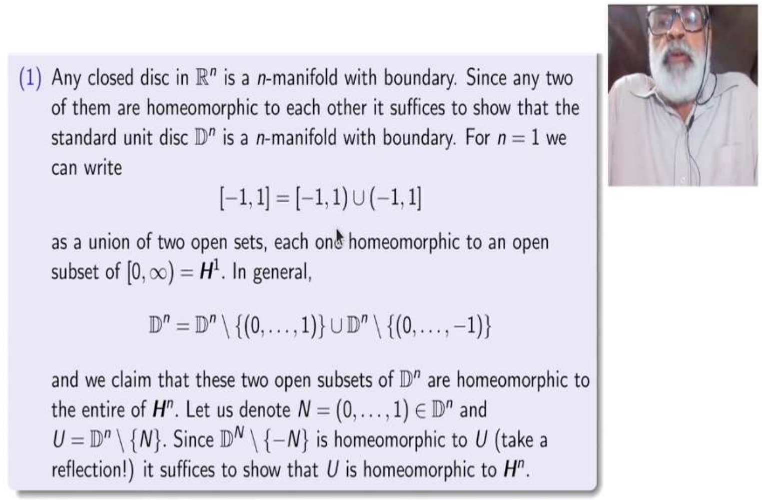

Let us have some examples.

Any closed disc in $\mathbb{R}^n$ is a manifold with boundary. So, we have to prove that one. Since any two of them are homoeomorphic to each other it suffices to show that the standard unit disk $\mathbb{D}^n$ inside $\mathbb{R}^n$ is an n-manifold with boundary. In fact boundary itself will be an $n-1$ dimensional manifold, all x such that norm $x = 1$. We will prove that. Let us look at the case $n = 1$ first. Then this closed disc is nothing but $[-1, 1]$, the interval.

You can write it as a union of $[-1, 1)$ and $(-1, 1]$. Two open subsets half closed intervals, each of them is clearly homoeomorphic to $[0, \infty)$ and that is $\boldsymbol{H}^1$. So, I have given an atlas consisting of two charts. So, I have given atlas over. In the general case what I will do? I will take $\mathbb{D}^n$ setminus the north pole $(0, 0, \dots, 0, 1)$ and $\mathbb{D}^n$ setminus the bottom point $(0, 0, \dots, 0, -1)$.

So, this is similar to what I did in the case $n = 1$. We claim that these two open subsets of $\mathbb{D}^n$ are homoeomorphic to the entire of $\boldsymbol{H}^n$ just like this one. dimensional version that is all. So, let us denote by a shorter notation this north by, say by N.

Let us consider one of them, $U = \mathbb{D}^n \setminus \{N\}$. The other one is homoeomorphic to this one by just taking the negative of the n^{th} coordinate, viz., reflection in the $(n-1)$ coordinate subspace, perpendicular to N. So, it is enough to show that this U is homoeomorphic to $\boldsymbol{H}^n$.

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First consider the map $f : \mathbb{D}^n \to \mathbb{S}^n$ given by

$$f(x_1, \dots, x_n) = \left(\sqrt{1 - \sum_i x_i^2}, x_1, \dots, x_n\right) =: (y_1, y_2, \dots, y_{n+1}).$$

Put $f(N) = (0, \dots, 0, 1) = N'$, say. It follows that f defines homeomorphism of $\mathbb{D}^n \setminus \{N\}$ onto the subset $B \setminus \{N'\}$ where

$$B = \{y \in \mathbb{S}^n \ : \ y_1 \geq 0.\}$$

Now if $\phi : \mathbb{S}^n \setminus \{N'\} \to \mathbb{R}^n$ is the stereographic projection then ϕ restricts to a homeomorphism of $B \setminus \{N'\}$ onto the space

$$H = \{(y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} \ : \ y_1 \geq 0, y_{n+1} = 0\}.$$

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$$H = \{(y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} \ : \ y_1 \geq 0, y_{n+1} = 0\}.$$

It is easily seen that H is homeomorphic to $\boldsymbol{H}^n$.

So, we have a ready made map here actually, only thing is we have to see how to use it. So, first what I do? I will identify this disc $\mathbb{D}^n$ with a subspace of $\mathbb{R}^{n+1}$ indeed a subspace of $\mathbb{S}^n$ itself. See $\mathbb{S}^n$ itself is sitting $\mathbb{R}^{n+1}$, this $\mathbb{D}^n$ is inside $\mathbb{R}^{n+1}$. So, I need 1 more coordinate. So take $(x_1, \dots, x_n)$ mapsto $(x_0, x_1, \dots, x_n)$ where the 0^{th}-coordinate x_0 is, first take $1 - \sum x_i^2$, and take the square root. Why I have done this? If you take now squares of all these coordinates now, that will be equal to 1. So the image point is inside the unit sphere here in $\mathbb{R}^{n+1}$. That is all.

Clearly this is a continuous map, it is actually the graph of this function x_0 and hence a homeomorphism onto its image viz., the closed upper hemisphere, the subspace of $\mathbb{S}^n$, given by the condition $x_0 \geq 0$.

So, this is the preparation I have made now. Now, what I do I take the stereographic projection. Stereographic projection is defined from $\mathbb{S}^n \setminus \{N'\}$ the north pole in $\mathbb{S}^n$ to the entire of $\mathbb{R}^n$.

So, that is a homeomorphism we have studied carefully. Restricted to the hemisphere setminus $\{N'\}$, it will go into the half space $\{(y_1, y_2, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} : y_1 \geq 0 \text{ and } y_{n+1} = 0\}$. This latter space is clearly homoeomorphic to $\boldsymbol{H}^n$, the only thing is instead of last coordinate $y_n \geq 0$, I put $y_1 \geq 0$. So, you have to interchange those two coordinates.

So, look at this method. I mean these are important methods in handling various subsets of $\mathbb{R}^n$, that is all.

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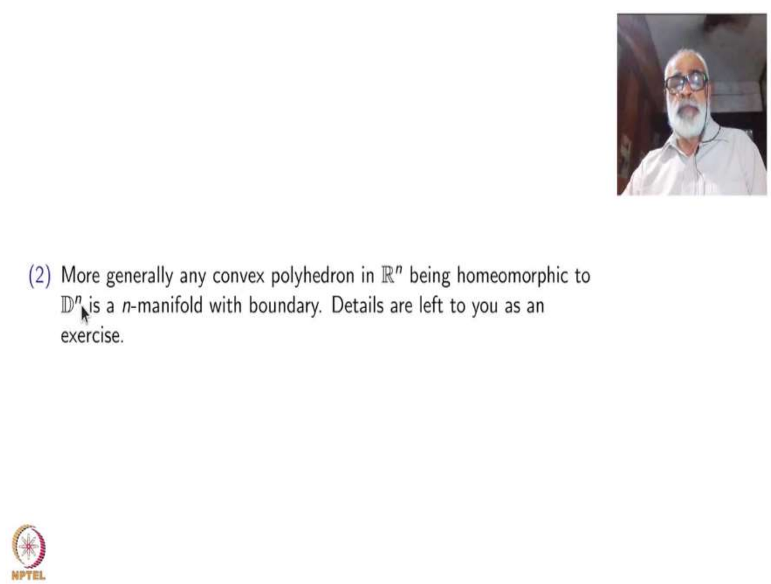

So, more generally, any convex polyhedron in $\mathbb{R}^n$ being homoeomorphic to $\mathbb{D}^n$ is a manifold with boundary. Details are left to you as an exercise, why? I do not want to get into what is the definition of convex polyhedron on and so on. If you do not know it, you will have to learn it from somewhere else that is all. That is not central to the theme of this course.

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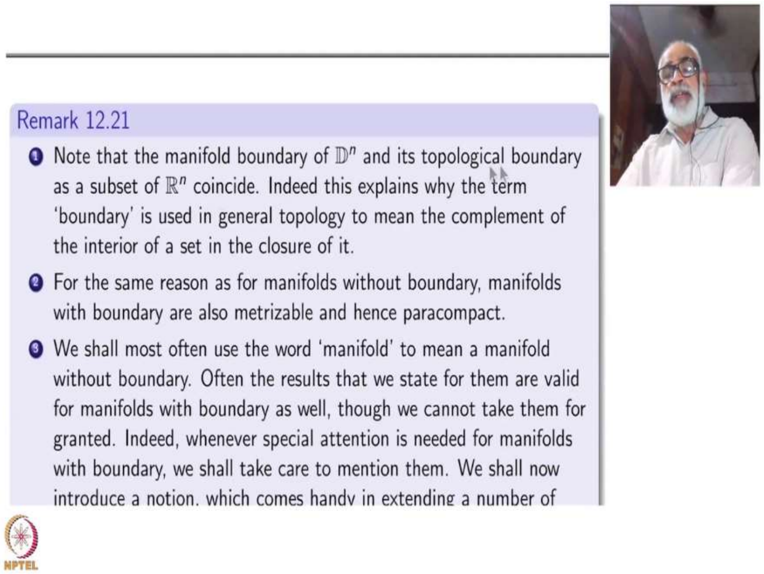

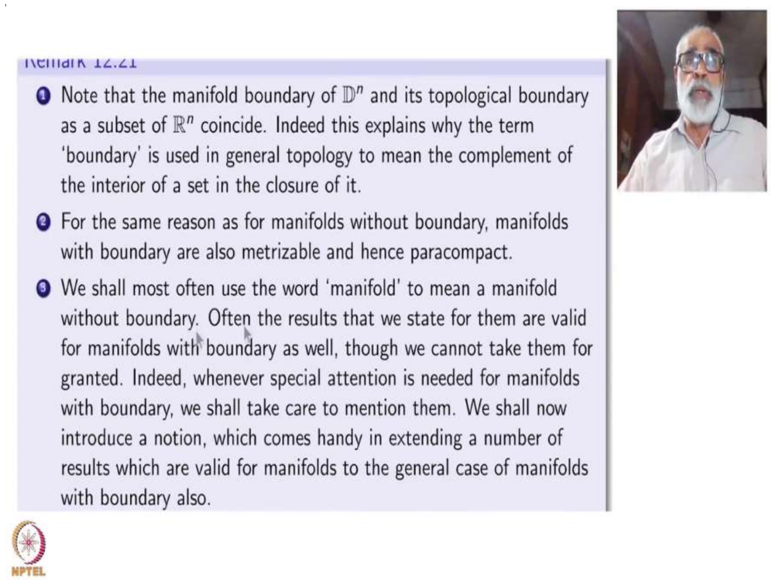

Note that as a manifold with boundary, the boudary of $\mathbb{D}^n$ coincides with its boundary as a subset of $\mathbb{R}^n$. Indeed this explains why the term boundary is used in general topology you know classically these manifolds were defined and studied even without definition perhaps even before the point-set-topology was conceived. Forget about the boundary of a subset and so on. So boundary of subset that concept is copied from examples like this. Unfortunately we are learning these things other way round. For the same reason as for manifolds without boundary, manifolds with boundary are also metrizable and hence paracompact, why?

Because we have assumed that they are all II-countable Hausdorff and whether they are locally homeomorphic to open subsets of $\mathbb{R}^n$ or $\boldsymbol{H}^n$, they will be locally compact. So, they are T_3 spaces. II-countable T_3 spaces are metrizable.

We shall often use the word manifold to mean manifold without boundary, the old definition.

So, that is why this shorter term manifold. Often the results that we state for them are valid for manifolds with boundary as well, though we cannot take all of them for granted there are some results which are not true at all for them. And if they are true, you have to prove them separately. Sometimes it requires quite a lot of effort as compared to manifolds without boundary. Whenever things are not true at all, we will try to mention them separately.

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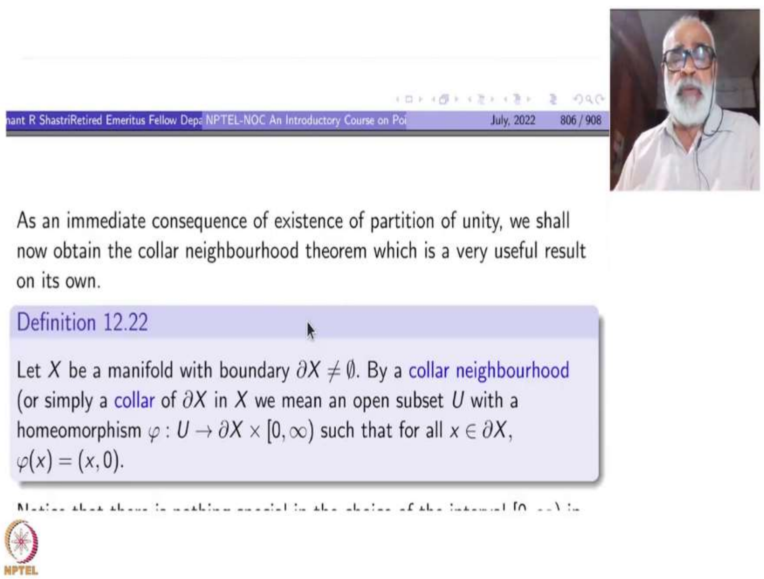

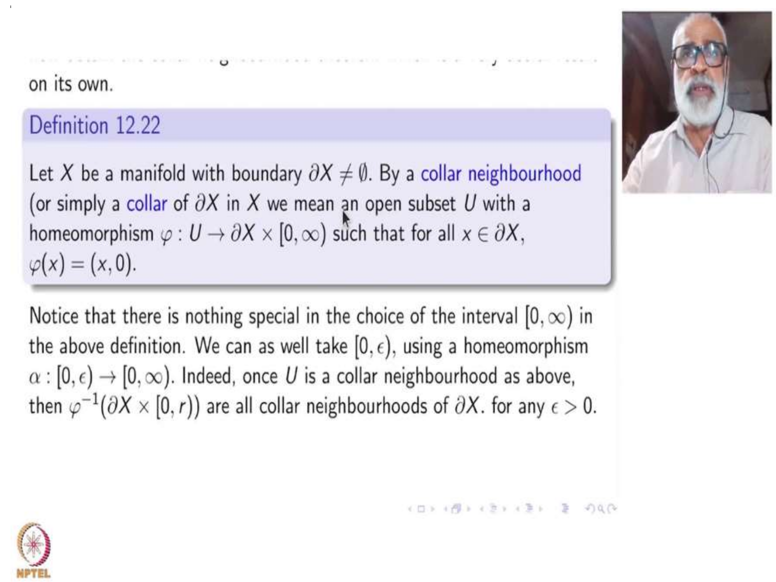

So here is an easy consequence of the existence of partition of unity because we have shown that they are Hausdorff and paracompact.

We shall now obtain the so called Collar neighbourhood theorem which is a very useful result on its own. So, that is an important result about manifolds with boundary.

So, let me make a definition first. Let X be a manifold with non empty boundary (otherwise the rest of the definition is vacuously true or it does not hold at all). So, start with the manifold with boundary non empty. By a `collar neighbourhood' (or simply you can say a `collar') of boundary of X inside X, we mean an open set U of X with a homeomorphism phi from U to the boundary of $X \times [0, \infty)$.

This is a topological product, boundary X is subspace of X, you must have a homeomorphism as above, with one extra property namely for all x in the boundary of X, $\phi(x)$ should go to $(x, 0)$. Clearly U a neighbourhood of boundary of X. So, boundary of X is sitting inside here. So, such an open neighbourhood along with a homeomorphism as above will be called a collar neighbourhood of boundary of X.

Now there is nothing special about the choice of this half open interval $[0, \infty)$, you could have taken $[0, \epsilon)$ for any epsilon positive, because we know $[0, \epsilon)$ is also homoeomorphic $[0, \infty)$ that is all. Indeed once (U, ϕ) is a collar neighbourhood as above you look at the subspace namely, $\phi^{-1}(\partial X \times [0, r))$, take the inverse image, that will be another open subset which will contain boundary of X, and it will be again homoeomorphic to $\partial X \times [0, infty)$ obviously, by composing ϕ with Id cross a homeomorphism from $[0, r)$ to $[0, \infty)$.

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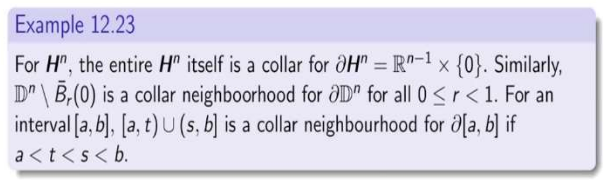

Note that the entire $\boldsymbol{H}^n$ itself is a collar for boundary of $\boldsymbol{H}^n$. Obviously, $\boldsymbol{H}^n$ is a manifold with boundary equal to $\mathbb{R}^{n-1} \times 0$.

Similarly, if you take $\mathbb{D}^n$ setminus a concentric closed ball, viz., the its centre is the origin and radius $r < 1$ of course, if you delete a concentric ball, then what you get is a collar neighbourhood for boundary of $\mathbb{D}^n$, namely, of the sphere $\mathbb{S}^{n-1}$.

For a closed interval $[a, b] \cup (s, b]$ is a collar neighbourhood of boundary of $[a, b]$, (if you choose $a < t < (b - a)/2$ and $s = b + a - t$). What is boundary of $[a, b]$? It is $\{a, b\}$ just the two elements, with the discrete topology. That has a neighbourhood inside $[a, b]$, like this $[a, t) \cup (s, b]$ (which homeomorphic to a product provided they are disjoint and of the same length).

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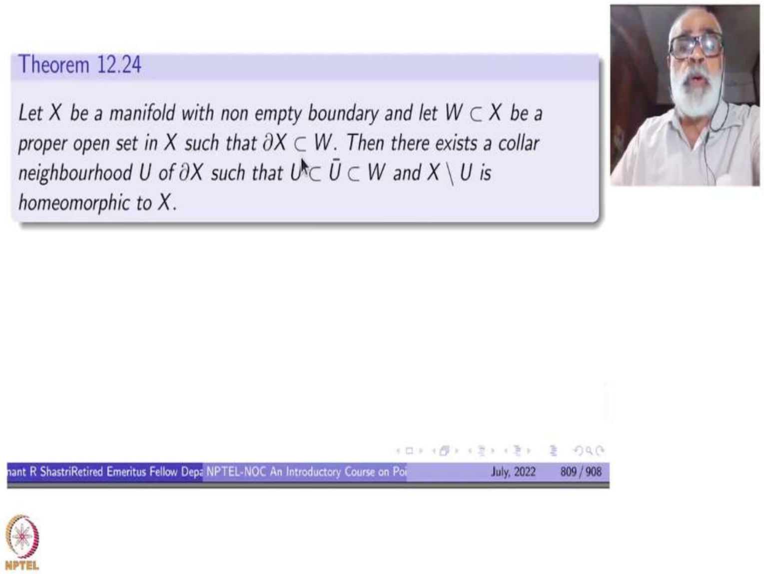

So, there is a final example now. Here it is stated as a theorem. So, pay attention.

Let X be a manifold with non-empty boundary and W contained in X be proper open subset in X such that boundary of X is contained inside W. I do not want this W to be the whole of X. Then there exists a collar neighbourhood U of boundary of X such that this U is contained in $\bar{U}$ contained inside W, and the complement $X \setminus U$ is homoeomorphic to X again.

So, throw away the collar neighbourhood. Whatever left out is again homoeomorphic to the manifold itself. The collar is always like boundary cross an interval. So, you remove it, that will be again homoeomorphic to X. That is the meaning of this one here. And such neighbourhood can be chosen as small as you please, viz., inside any given open set W. This W itself is a neighbourhood of boundary of X. Then you can take U such that $\bar{U}$ inside W.

So, the proof is slightly longer but not difficult if you understand what is going on. The two, three examples whatever I have given above, they are the guiding principles here. That is all.

Look at what happens in that place of 0 to say 1 open. $\{0\}$ is the boundary it has a neighbourhood no matter how small ϵ you take $[0, \epsilon)$, which is homomorphic to $[0, \infty)$. If you remove $[0, \epsilon)$ whatever is left is again a half closed interval and so it will be again homeomorphic to the whole of $[0, \infty)$.

Similar to that is what is happening. But we have to work a little harder here, we need to use partition of unity also. Because we are not assuming compactness. If you assume compactness, you can probably write down a much shorter proof. Over to you, after you learn this proof that will be left as an exercise to you.

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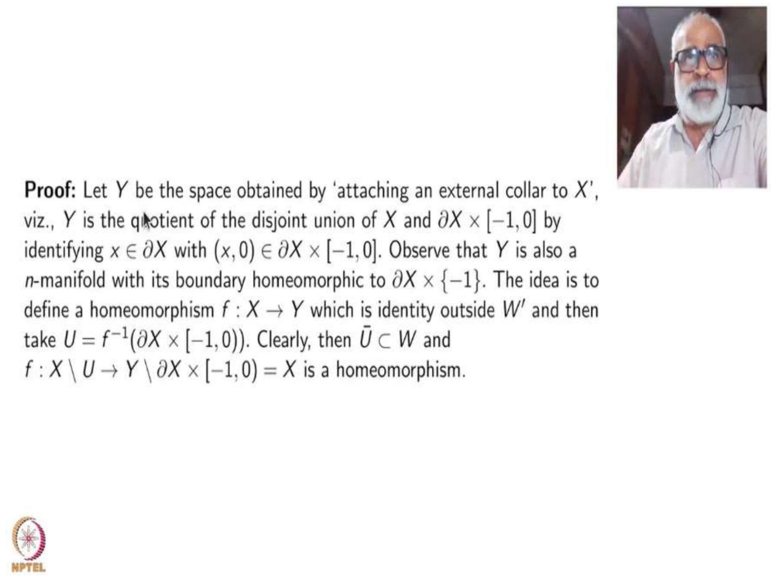

So the idea here is a beautiful one here, what we do is, we attach an external collar to X, I am going to explain what is the meaning of this one. Namely, you know, you are taking a larger space, you are creating a space containing X. So the rest of the extra space is boundary of X cross $[-1, 0]$.

So that is what I am going to do. So, I am taking Y as the quotient of disjoint union of X and boundary of $X \times [-1, 0]$, (if you just take disjoint union that is not much fun, what I do?) where I identify boundary of $X \times \{0\}$, this copy of boundary of X, with actual boundary of X here, by identifying $(x, 0)$ with x for every $x \in \partial X$. The rest of the points of the disjoint union are floating outside undisturbed. Observe that Y is also a n-manifold with its boundary homeomorphic to boundary of $X \times \{-1\}$ this time. So, this portion becomes the boundary now. All the points $(x, 0)$, they have become interior points in Y.

The idea is to define a homeomorphism f from X to Y, which is identity outside some W' properly chosen and then take U equal to $f^{-1}(\partial X \times [-1, 0)) \in Y$. Clearly then $\overline{U}$ will be contained inside W and f from here to here is a homeomorphism.

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By regularity, can choose an open set W' such that $\partial X \subset W' \subset \bar{W}' \subset W$. Now, begin with a (countable) partition of unity $\{\theta_i\}$ on ∂X so that for each i $\mathrm{supp}\,\theta_i$ is compact and is contained in a coordinate open set U_i of ∂X together with a homeomorphism $\phi_i : U_i \times [0,1) \to V_i$ onto an open subset $V_i \subset W'$ of X such that $\phi_i(x,0) = x$ for all $x \in U_i$. Put $\eta_0 = 0, \eta_k = \sum_{i=1}^{k} \theta_i, k \geq 1$; and

$$Z_k := \{(x,t) \; : \; x \in U_k, -\eta_{k-1}(x) \leq t \leq 1\};$$

$$Z_k' = \{(x,t) \; : \; x \in U_k, \; -\eta_k(x) \leq t \leq 1\}.$$

Proof: Let Y be the space obtained by 'attaching an external collar to X', viz., Y is the quotient of the disjoint union of X and $\partial X \times [-1,0]$ by identifying $x \in \partial X$ with $(x,0) \in \partial X \times [-1,0]$. Observe that Y is also a n-manifold with its boundary homeomorphic to $\partial X \times \{-1\}$. The idea is to define a homeomorphism $f : X \to Y$ which is identity outside W' and then take $U = f^{-1}(\partial X \times [-1,0))$. Clearly, then $\bar{U} \subset W$ and $f : X \setminus U \to Y \setminus \partial X \times [-1,0) = X$ is a homeomorphism.

So, what is this W', I will explain. This is a smaller subspace of W chosen right in the beginning like this. Namely, you can choose an open subset W' such that boundary of X is contained inside W' and its closure is contained inside W. So, here I am just using regularity and ∂X is a closed subset. Closed subset contained in open subset, in between you can introduce this another closed neighbourhood.

So, now we begin with a countable partition of unity $\{\theta_i\}$, how did you get countability? Because the whole space is II-countable. Where I am working, on the boundary of X which is a manifold.

So that for each i, the support of θ_i is compact. So, this is one of the remark which we have made. Because of the local compactness, you can assume that the support of θ_i are compact, and is contained in a coordinate open subset U_i. The coordinate subsets can be assumed to be such that they are relatively compact, their closures are compact. And then if this support of θ_i is contained inside U_i then automatically, the support will be also compact, being a closed subset.

So, how do I do that? Start with an open cover consisting of coordinate neighbourhoods, namely an atlas. Go to a locally finite subcover, possible because ∂X is paracompact. After that go to a countable subcover, possible because of II-countability. Use the partition of unity subordinate this final cover, which exists because of paracompactness. Automatically you will get the supports to be compact.

So, these are coordinate neighbourhoods. So we have ϕ_i from $U_i \times [0, 1)$ to V_i, where each V_i is open subset of W'. These are coordinate neighbourhoods for points inside boundary of X. Now I am going inside the manifold X itself, by using the fact that whole thing W' is a neighbourhood of boundary of X.

And $\overline{U_i}$ are compact. So I can extend this neighbourhood to $[0, 1)$, V_i from $U_i \times 0$. See you the first thing here because of the compactness, you can extend it to a small neighbourhood 0 to 1 to V_i is Weyl's theorem. Homeomorphism V_i to some $\mathbb{R}^n$ is there.

Now, $\mathbb{R}^{n-1}$ is there now you can take an open subset in $\mathbb{R}^n$, open subset V_i's, these V_i's are inside W' of X. Such that the if ϕ_i, 0^{th} coordinate $(x, 0)$ is $x \in V_i'$ for all x. So, these U_i's. ϕ_i's I am actually writing as a parameterization there you know parameterizations for points inside the boundary, but the parameterization occurring for neighbourhoods inside X itself. So, you choose such homeomorphism that is no problem.

Now comes the inductive construction. Now, put η_0 equal to 0. So, starting with not doing anything η_0 equal to 0. For $k > 0,\ \eta_k$ is the sum of all θ_i for i ranging from 1 to k. Remember, these θ_i are all non negative real valued functions.

So, you can just sum up finite number of them, that is the definition of these new fuctions η_k. Next, put this Z_k equal to the space of all (x, t) such that x is inside U_k, (they are open subsets of ∂X) and t varies between $-\eta_{k-1}(x)$ and 1. After all, these η_k's are taking values between 0 and 1. So I take minus of this less than equal to t less than equal to 1 these is the range for the second coordinate. So this is some interval strictly contained inside of $[-1, 1]$. This part is 1 on the positive side, the negative side how much you can go at the most -1. So, these are subspaces of $U_k \times [-1, 1]$, but may be much smaller.

Now put Z_k' equal to the space of all (x, t) such that similar to Z_k, but use η_k here instead η_{k-1}. So, since η_k is slightly bigger that η_{k-1} because one more function has been added here, this interval will be slightly larger interval than this one but the first coordinate is the same U_k.

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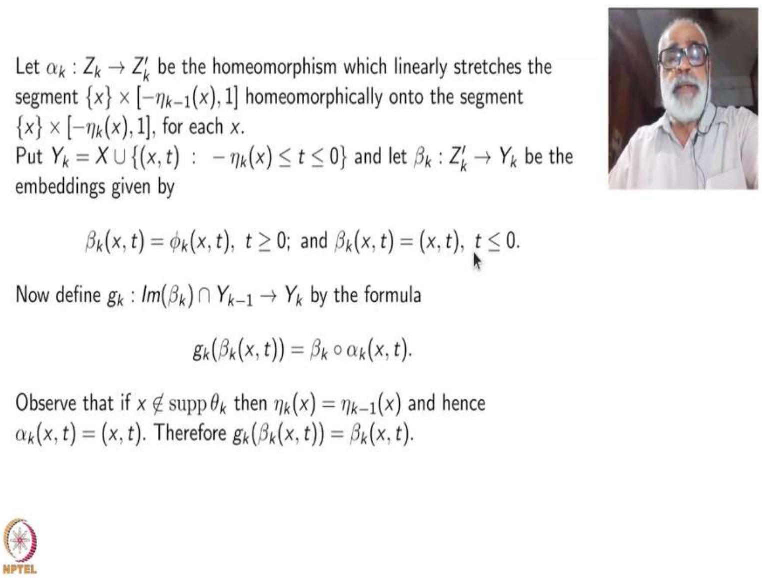

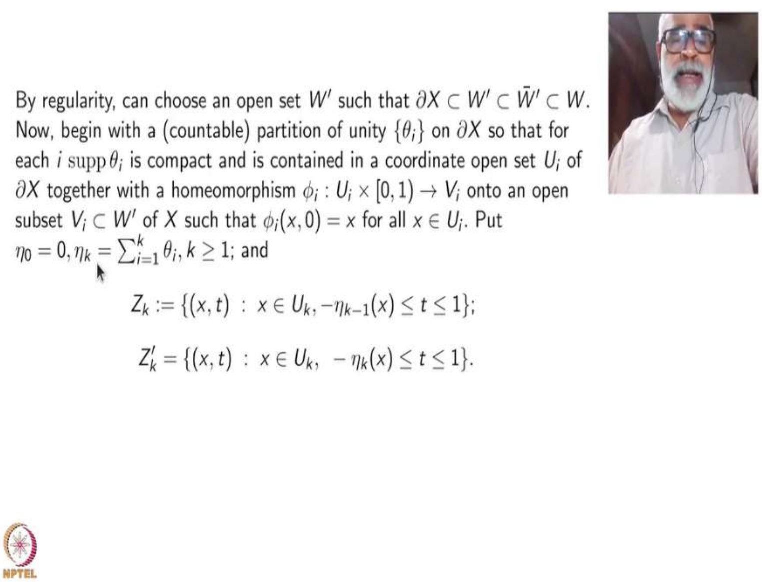

And now, take α_k from Z_k to Z_k' be the homeomorphism which linearly stretches the segment $\{x\}$ cross this interval to this interval, linearly stretches means what? 1 goes to 1 and$-\eta_{k-1}(x)$ goes to $-\eta_k(x)$. You know, we have a unique linear order preserving homeomorphism mapping any interval $[a, b]$ to $[a', b']$. So, take those homomorphisms for each x you do that. Because you can write down the formula for them in terms of in terms $\eta_{k-1}(x)$ and $\eta_k(x)$, it will be automatically a homeomorphism on the whole space Z_k.

So, put Y_k, (I am defining these things inductively), put Y_k equal X union the set of all $\{(x,t) : \eta_k(x) \text{ is taken up to } 0 \text{ only}\}$. So, this the portion Z'_k taken only up to $t \leq 0$. These Y_k are extra spaces you see that is what I have taken union with X.

So, they are all subspaces of Y now anyway. Let β_k from Z'_k to Y_k be the embedding given by $\beta_k(x,t)$ is $\phi_k(x,t)$, where t is positive, sorry non negative. (Remember this ϕ_k is defined on $U_k \times [0,1]$), so, when the second coordinate is no negative here, you take β_k to be ϕ_k; when t is between -1 and 0, you take $\beta_k(x,t) = (x,t)$ itself.

So, that makes sense because they are all subspaces of Y. We are working inside Y now. So, it will go inside Y_k. β_k is well defined, because though it is given by two definitions for $t = 0$, the two definitions coincide. So, we have got these embedding of Z'_k inside Y_k.

Next, we define g_k on the image of $\beta_k \cap Y_{k-1}$ to Y_k by the formula $g_k(\beta_k(x,t)) = \beta_k \circ \alpha_k(x,t)$. This α_k is stretching the smaller space to the larger space.

So, first stretch it and then take β_k, every member here is the image of a unique point, it is $\beta_k(x,t)$. So, take that (x,t) apply the stretching, and then again take β_k. So it is like first you take β_k^{-1} of the point, to begin with. Instead of writing inverse I have just written it as $g_k(\beta_k(x,t))$. Every element here in the image of β_k, is after all the image of a unique element like this $\beta_k(x,t)$

If x is not in the support of θ_k, i.e. $\theta_k(x) = 0$, then $\eta_k(x)$ will be $\eta_{k-1}(x)$, and so the stretching fact is identity. Therefore, $\alpha_k(x,t)$ will be just (x,t), therefore, $g_k(\beta_k(x,t))$ will be $\beta_k(x,t)$ for such points. For negative t, anyway $\beta_k(x,t)$ is (x,t). But now we have seen that of this one $g_k(\beta_k(x,t))$ will be just $\beta_k(x,t)$. Because this is identity.

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Now define $f_k : Y_{k-1} \to Y_k$ to be g_k on $Im\beta_k$ and Id on $Y_{k-1} \setminus Im\beta_k$. It follows that f_k is continuous. Indeed since each α_k is a homeomorphism, it easily follows that f_k is a homeomorphism.
Finally put $f = \cdots \circ f_k \circ f_{k-1} \circ \cdots \circ f_1$.
First of all note that on the complement of $V = \cup_i \phi_i(U_i \times [0,1))$, and hence outside W', f is identity. On V itself, f makes sense, since given any point $x \in \partial X$, there are only finitely many i for which $x \in U_i$ and $f_k(x,t) = (x,t)$ if $x \notin U_k$. Indeed in a neighbourhood of x, all f_k are identity except those k for which $\theta_k(x) \neq 0$.

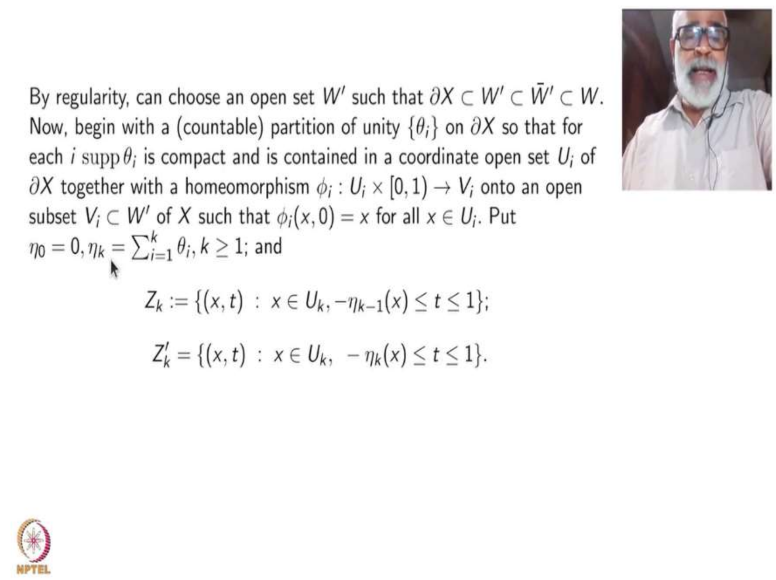
Let $\alpha_k : Z_k \to Z'_k$ be the homeomorphism which linearly stretches the segment $\{x\} \times [-\eta_{k-1}(x), 1]$ homeomorphically onto the segment $\{x\} \times [-\eta_k(x), 1]$, for each x.
Put $Y_k = X \cup \{(x,t) \ : \ -\eta_k(x) \leq t \leq 0\}$ and let $\beta_k : Z'_k \to Y_k$ be the embeddings given by

$$\beta_k(x,t) = \phi_k(x,t), \ t \geq 0; \text{ and } \beta_k(x,t) = (x,t), \ t \leq 0.$$

Now define $g_k : Im(\beta_k) \cap Y_{k-1} \to Y_k$ by the formula

$$g_k(\beta_k(x,t)) = \beta_k \circ \alpha_k(x,t).$$

Observe that if $x \notin \operatorname{supp} \theta_k$ then $\eta_k(x) = \eta_{k-1}(x)$ and hence $\alpha_k(x,t) = (x,t)$. Therefore $g_k(\beta_k(x,t)) = \beta_k(x,t)$.



So, these stretching are extending the earlier neighbourhoods to the next one. That is important. Now, we define f_k from Y_{k-1} to Y_k, (see on these subspaces this β_k, we have defined already, now on the whole space Y_k) to be g_k on image of β_k and identity outside the image of β_k.

On the boundary of image of β_k itself g_k is identity, therefore I can extend it identically on outside of image of β_k, it will follow that f_k is continuous.

Indeed, since each α_k is a homeomorphism, it easily follows that f_k is also a homeomorphism. Finally, there are infinitely many of these f_k's. I am putting f equal to the composition of these f_k ’s, in that correct order, the reverse order. So, what is the meaning of this? How do you explain this infinite composition? Look at any point, f_1 is defined, f_2 is defined, f_3 and so on. After a certain stage it will become identity Why? Once the point is outside support of θ_k, f_k will be just identity.

So, we are taking only the composition up till here, stretching, stretching, stretching along the these intervals, vertical intervals over x. x will be fixed and (x,t) goes to (x,t'), that is the kind of homeomorphism we have. So, this makes sense because at each point only finitely many functions are not identity after that all are identity. So, this is like infinite product of real numbers, when you take most of them equal to 1. Similar to that.

So, note that on complement of V which is the union of all $\phi_i(U_i \times [0,1))$, which is outside W', f is identity. On V itself, f makes sense. Since for any point x belonging to boundary of X, there are only finitely many i for which x belongs to U_i because of locally finiteness. And $f_k(x,t)$ is equal to (x, t), if x is not in U_k, x is not in the support of θ_k. Indeed in a neighbourhood of x, all f_k are identity except those k for which $\theta_k(x)$ is not 0. So that is the whole idea.

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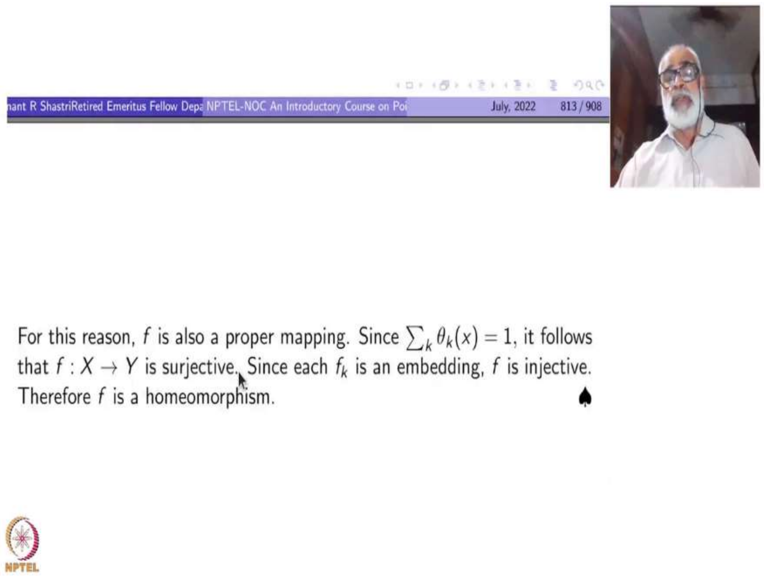

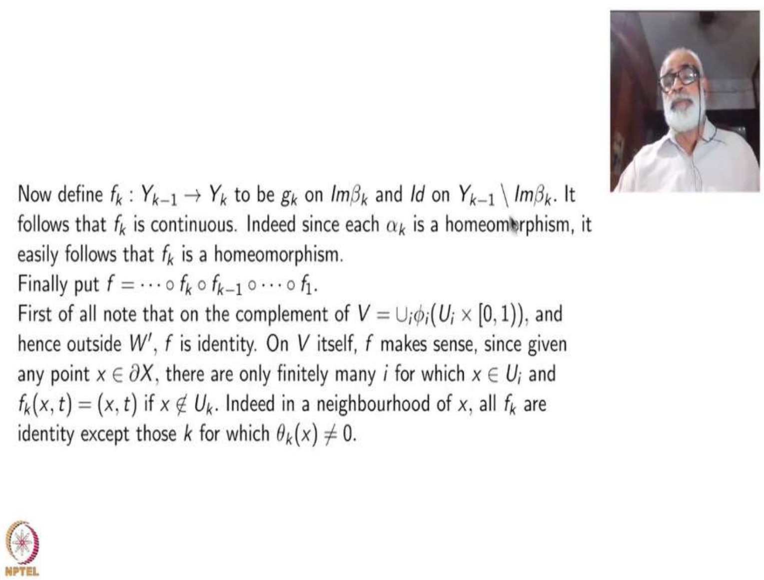

For this reason, the map f is also a proper map, why? Enounh to prove that inverse image of a compact subset $K \times [0, b]$ for each $0 < b < 1$, is compact. Such an inverse image again contained inside $K \times [0, b']$ for some b' between 0 and 1 is what you have to see. This follows because for each such K, you can find a k such that f_i are identity outside K, for $i > k$.

Finally, since summation $\theta_k(x) = 1$ for all x, it follows that f will be surjective. Though it is infinite sum in as such given any x, there will be only finite many i such that $\theta_i(x)$ is not zero, therefore if $t < 1$, there will be some k such that t lies between $\eta_{k-1}(x)$ and $\eta_k(x)$ and hence (x, t) will be hit by f_k.

So that is why the surjective.

Since each f_k is an embedding, f is injective also, therefore, f is a homeomorphism being a proper injective and surjective map.

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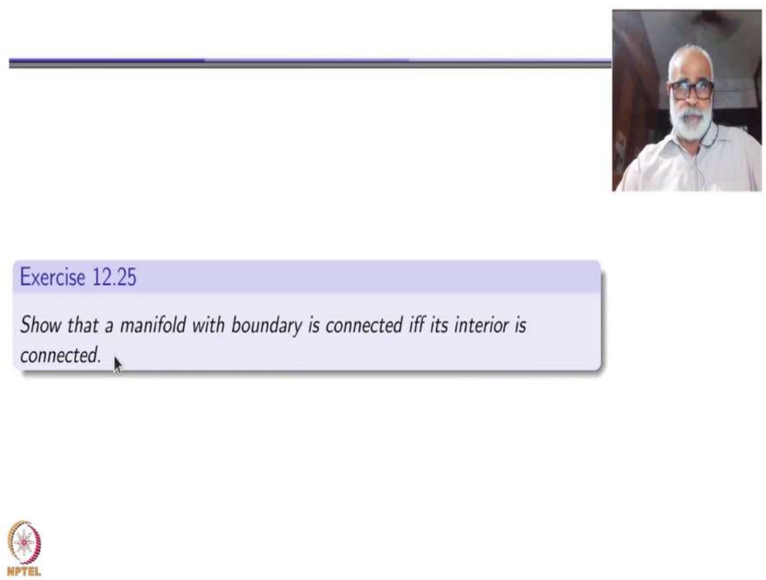

Here is an easy exercise. Show that a manifold with boundary is connected if and only if its interior is connected. This is slightly tricky thing. You have to think about this, interior is connected of course interior is connected but what about the points on the boundary. Why the whole thing is connected that is what you have to show. But that will make you think what exactly you have to use here. So next time we will do some more properties of manifolds. Mostly we will now deal with only manifolds, that is, manifolds without boundary. Thank you.