An introduction to Point-Set-Topology Part-II
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Week 11
Lecture 53
An Application to Quotient Maps

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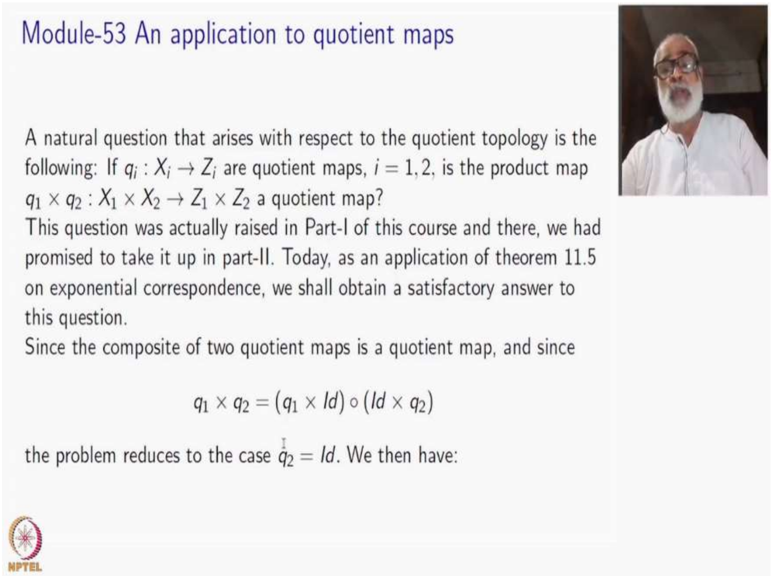

Hello, welcome to module 53 of NPTEL NOC on point-set-topology part II. The title of this today's talk is an application to quotient maps and of the exponential correspondence that we have studied last time. A natural question that arises with respect to the quotient topology is the following.

Suppose, you have two quotient maps q_i from X_i to Z_i, i equal to 1 and 2, is the product map $q_1 \times q_2$ from $X_1 \times X_2$ to $Z_1 \times Z_2$ a quotient map? So, you may be surprised that, in the general this is not true. See surjectivity will be still there, continuity still there. That is all. If q_1 and q_2 are open maps the product is also an open map and hence quotient map. But in general, we have to study other quotient maps, not all quotient maps are open maps.

So, here is one satisfactory answer. In any case, this question was actually raised in part I itself when we were studying quotient maps. So, today we will have a satisfactory answer.

In general, $q_1 \times q_2$ can be written as q_1 cross identity composed with identity cross q_2. This is just a set theoretic fact and nothing to do with quotient maps. So, if I can shown that each one on the right hand side is a quotient map, composite of quotient maps is a quotient map and hencw $q_1 \times q_2$ will be also quotient map.

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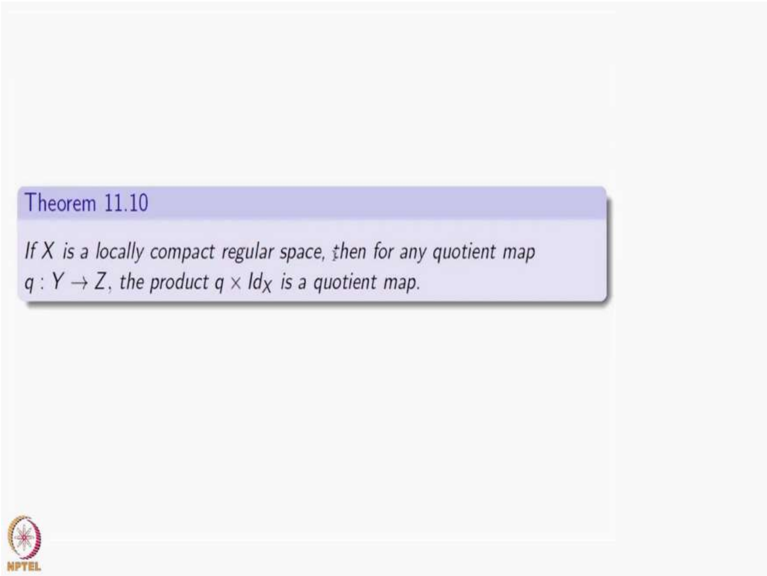

Therefore, question is reduced to the special case when out of q_1 and q_2, we assume one of them is an identity map. By symmetry, whichever one we assume is identity map, it is the same thing. q_1 cross identity or identity cross q_2. If we show that for all q_2, this is true that is fine. It is also fine if it is true for all q_1.

So, what we shall do? We shall show that if X is locally compact regular space, Then for any quotient map q from Y to Z, q cross identity of X is a quotient map. So, this is the condition on X, locally compact regular, or locally compact Hausdorff (which anyway implies locally compact regular).

(Refer Slide Time: 4:07)

Let us recall a fact about quotient topologies. A continuous surjective map of topological spaces is a quotient map, if and only if it satisfies the following property. Given α from A to B that is a continuous surjection, when is it a quotient map or when the topology on B will be a quotient topology? That is the condition for every topological space W, every function g from B to W some set theoretic function, if the composite $g \circ \alpha$ is continuous, then g must be continuous. g is continuous then the composite is continuous is obvious, because composite of continuous functions is continuous. Here it is the other way round. If this is satisfied for every g, then α will be a quotient map. So, I am not going to prove this one, this has been proved and used several times. And we will use it now.

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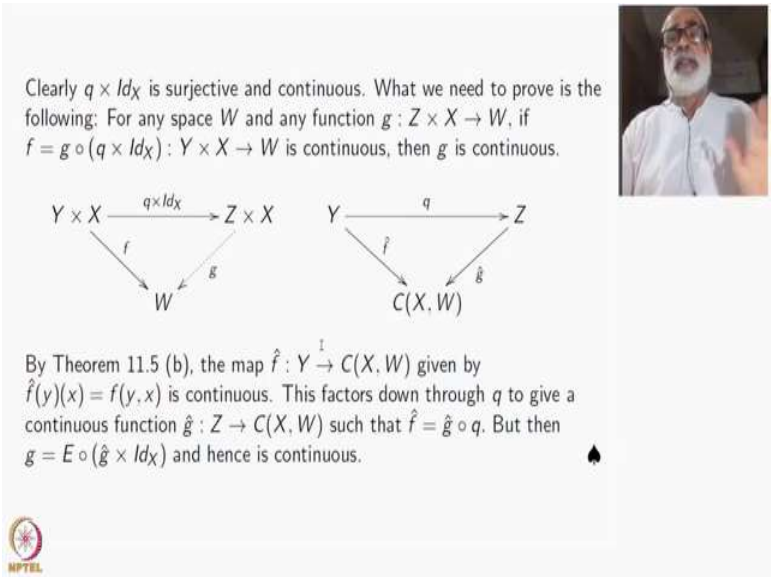

Starting with a quotient map q from Y to Z, under the hypothesis that the topological space X is locally compact and regular, we are going to prove that the product map $Id_X \times q$ is a quotient map. What we need to prove for this? We need to prove the following condition: Take any W, and g from $Z \times X$ to W, any function, such that when you compose it with q cross identity that is continuous, then g is continuous, this is what we have do.

So, this is the diagram: Y cross X to Z cross X, q cross identity of X, we want to prove this is a quotient map. So, take any W, and any g here, assuming that g composite, this q cross identity is continuous, we have to show that g that is continuous.

Now, how do you use the exponential correspondence here. When you have a continuous function f on $Y \times Z$ to W, we know this is the same thing as saying that there is a continuous function from Y to the space of all continuous functions from X to W. See, what is that map?

It is given by the exponential correspondence. This f corresponds to $\hat{f}$ here given by $\hat{f}$ of any $y \in Y$ operating upon $x \in X$ is $f(y, x)$. So, you get a continuous function. The exponential correspondence says that $\hat{f}$ is continuous if and only if f is continuous. So, this we are going to use now. So, once you have this continuous function f, we get $\hat{f}$ continuous. Similarly, for the function g, we get the corresponding function $\hat{g}$ from Z to $\mathcal{C}(X, W)$. I do not know whether it is continuous, but what I know is that $\hat{g} \circ q$ is $\hat{f}$ because, by definition, $\hat{g}(z)$ operation on any x is $g(z, x)$. Putting $z = q(y)$, we easily check that $g(q(y), x) = f(y, x)$. That means $\hat{g} \circ q$ is $\hat{f}$. Since $\hat{f}$ is continuous and q from Y to Z is a quotient map it follows that $\hat{g}$ is continuous. But once $\hat{g}$ is continuous, exponential correspondence says that this g is continuous. The proof is over.

So, we have not proved that product of any two quotient maps is a quotient map. That is not true. If one of them is locally compact Hausdorff, or locally compact regular then it holds. This is what will happen.

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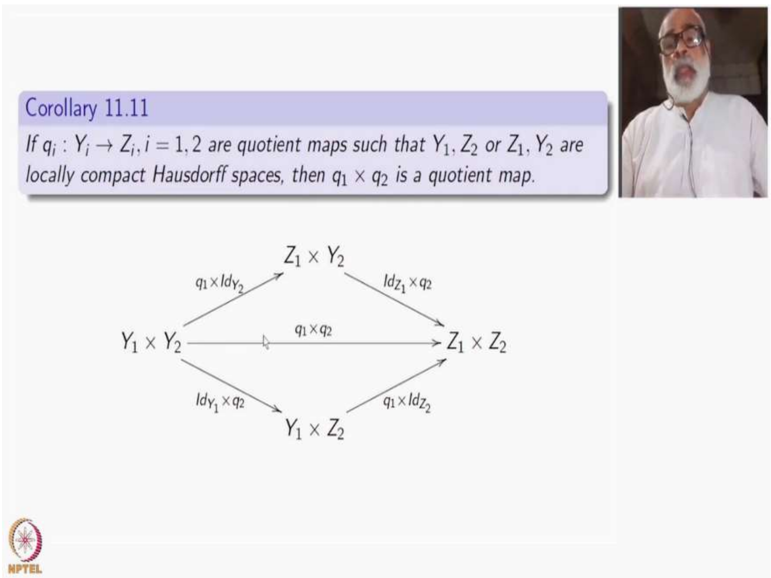

So, now we can use that to just give you satisfactory answer. This is not an `if and only if' kind of answer. But it is a useful thing. Suppose you have $Y_1 \times Y_2$ to $Z_1 \times Z_2$, $q_1 \times q_2$ a product of two quotient maps. From $Y_1 \times Y_2$ to $Z_1 \times Z_2$, in two different ways. q_1 cross identity of Y_2, then followed by identity of Z_1 cross q_2. Or first identity of Y_1 cross q_2, and then followed by q_1 cross identity of Z_2. So, the statement is that if Y_2 and Z_1 are locally compact regular, this composite is a quotient map because each of them is a quotient map.

Or you may use the other hypothesis here namely Z_1 and Y_2 are locally compact and regular . So, Y_1, Z_2, or Z_1, Y_2. There is no necessity that both should hold. Y_1 and then $q_1 \times q_2$ is a quotient map.

Or you may use the other hypothesis here namely Z_1, say Y_2 is a question map, Y_2 is local compact regular, and here Z_1 is locally compact regular. So, Y_1, Z_2, or Z_1, Y_2 or there is no necessity that both should be there. Y_1 and Z_2, or Z_1 and Y_2 that is the meaning of this one. Locally compact Hausdorff space. Then $q_1 \times q_2$ is a quotient.

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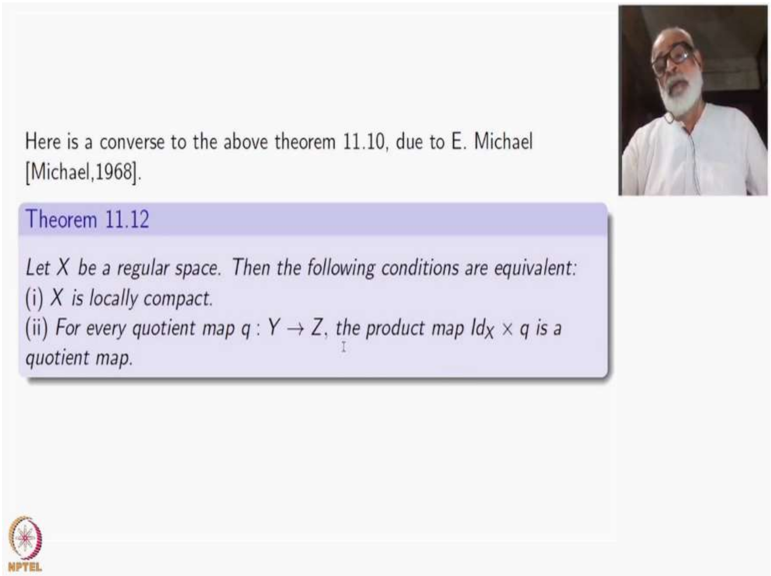

Now, we come to the main theorem in this section, in this today's talk and that is due to Michael. So, here is a partial converse to the above theorem. Start with a regular space X (or maybe you can start with a Hausdorff space, it does not matter, one of the hypothesis is important.) Then the following conditions are equivalent. So, it is only a partial converse, regularity cannot be replaced.

(i) X is locally compact.

(ii) Every quotient map q from Y to Z, the product identity of X cross q is also a quotient map.

(i) implies (ii) is what we have seen. (ii) implies (i) is the converse that you have to prove. So, if this happens viz., for every quotient map q from Y to Z, identity cross q is a quotient map, then X must be locally compact.

This is much stronger than just producing an example of a non locally compact space X but here this is a theorem, which says that for every non locally compact space this happens.

How do you prove this? By assuming that X is not locally compact, we will construct a space Y and a quotient map of Y to Z, such that identity of X cross q is not a quotient map.

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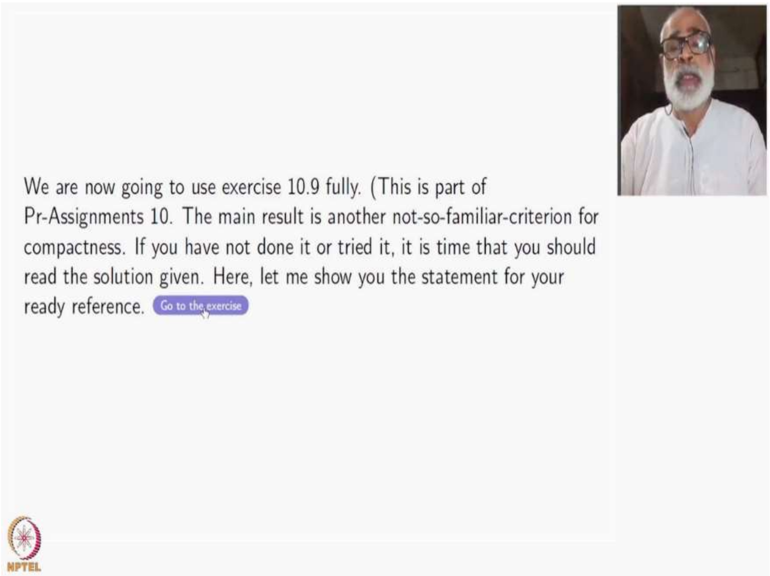

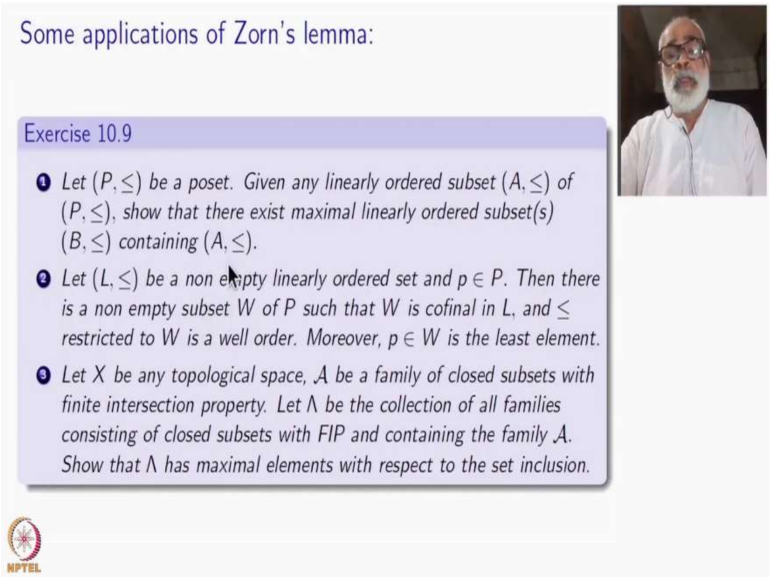

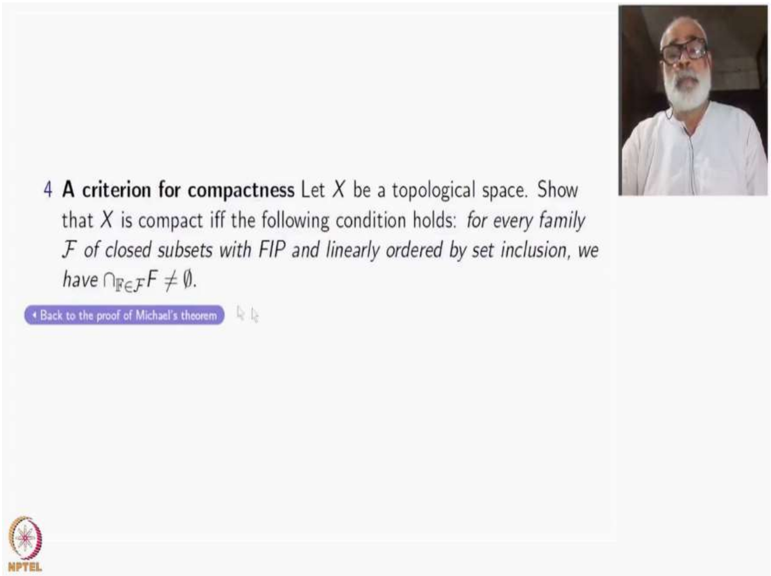

Here we are going to use the exercise 10.9 fully. This is part of your preparatory assignment or practice assignments 10. This was not in the main part of the assignment but in the practice session. The main result is this exercise is another not so familiar criterion for compactness. That is also part of this one. If you have not done it, or tried it, it is time that you should read the solution given.

However, right now, I cannot afford to go through that one. So, what I will do, I will just recall this exercise here and then use it. Look at this part (4) here. So, this is the main thing that I have to use. Let X be a topological space. Show that X is compact if and only if the following condition holds:

For every family $\mathcal{F}$ of closed subsets of X with finite intersection property, and linearly ordered by set inclusion, we have intersection of all members of this $\mathcal{F}$ is non empty.

Remember, if I remove this `linearly ordered by set inclusion' phrase, if I remove this condition, then this is a familiar theorem for you. Namely, if every family of closed subsets of X with finite intersection property has a non empty intersection then X is compact. So, that is a more general and weaker condition. So here we do not need that. Only those families which are linearly ordered by set inclusions are considered. (You can either take inclusion or reverse inclusion, it does not matter). This is somewhat similar to Cantor's interection theorem, a kind of converse. However, don't be mislead by this analogy, there is no countability assumption here. $\mathcal{F}$ is just a linearly ordered. That is the whole point here. Now, try to prove this one, but later on anyway, we will give you a solution.

The other thing is this problem (3) here. Let X be sorry, this one namely problem (2) here. Let $(L, \leq)$ be a non empty linearly ordered set with $p \in L$. Then there is a non empty subset W of this L such that W is cofinal in L, and restricted to W, this ordering itself is well ordered. Moreover, the point p that you have chosen is the least element of W.

So, I do not use in its full force. What I need is that every non empty linearly ordered set has a non empty well ordered subset which is (4) So, this is what I am going to use. To prove this one, you will need (1) To prove (4), you will need (3) also etc. So you better read on your own all the four $1, 2, 3, 4$.

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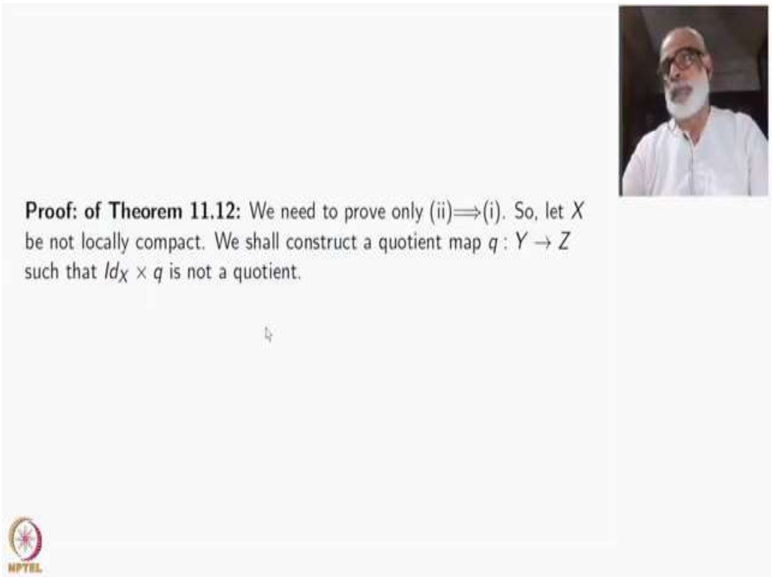

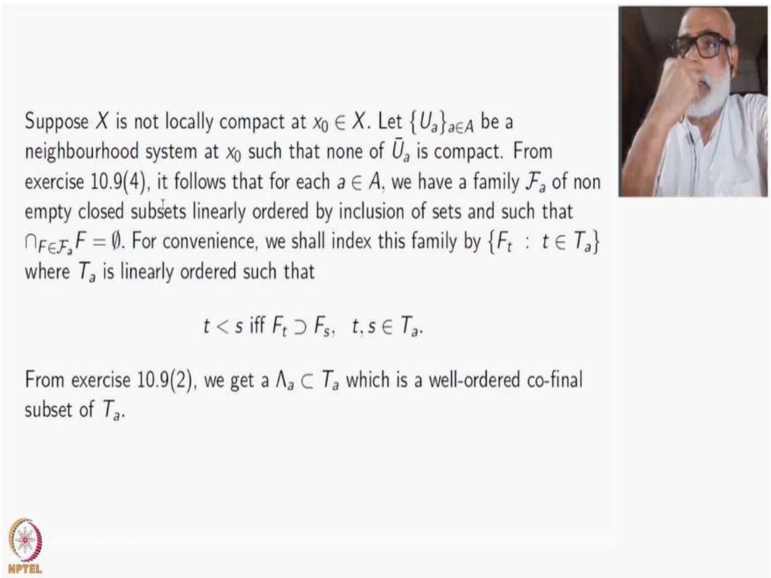

So, let us go back to Michael’s theorem. As I told you to prove the equivalence of (i) and (ii), we have only to prove (ii) implies (i). (i) implies (ii) has been taken care by the previous theorem.

So, start with a topological space X, which is not locally compact. We shall construct a quotient map q from Y to Z such that identity of X cross q, (or q cross identify, it does not matter) is not a quotient map.

Suppose X is not locally compact means that local compact fails at some point x_0 in X. What does that mean? There is a family $\{U_a, a \in A\}$ (this A is nothing but just an indexing set) such that $\{\overline{U_a}\}$ form a fundamental neighbourhood system of closed neighbourhoods at x_0 such that none of $\overline{U_a}$ is compact. So this is the meaning of that X is not locally compact at x_0.

Each $\overline{U_a}$ is not compact, what does that mean? Now, I am using this exercise 10.9(4). For each $a \in A$, we have family $\mathcal{F}_a$ of non empty closed subsets of U_a linearly ordered by inclusion of sets, such that intersection of all F, F belong to $\mathcal{F}_a$ is empty. Since it is linearly ordered, I do not have to mention finite intersection property, which is automatic, because I have said that $\mathcal{F}$ consists of non empty closed subsets. Linearly ordered by inclusion such that intersection is empty.

So, this is stronger than saying that there is some family with finite intersection property such that intersection of all members is empty. So, this is where we have used X of Y, 10.9(4).

Now for the sake of convenience of writing down the further proof, we index these families $\mathcal{F}_a$ as follows: $\mathcal{F}_a = \{F_t : t \in T_a\}$ and shift the linear order on $\mathcal{F}_a$ to the indexing set T_a by the rule: $t \leq s$, if and only if F_t contains F_s (reverse inclusion order).

So, you may say that each $\mathcal{F}_a$ is a family of decreasing non empty closed sets. I cannot say decreasing sequence because this may not be countable. T_a's may be countable, they may be finite, but we do not know, they need not be countable. That is all.

From 10.9(2) about linear orders, we get a subset Λ_a of T_a which is well ordered and cofinal subset of T_a. Given any member t of T_a, there will be some s in Λ_a which is bigger than t with respect to the order in T_a. That is meaning of cofinal family. So, these two things right now, I have used for the exercise. From now onwards, we are just using ordinary constructions here, quotient space and so on.

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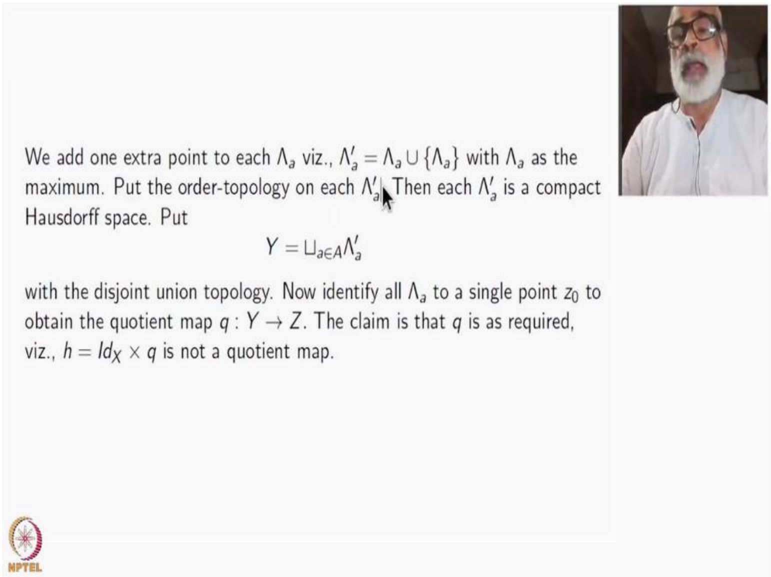

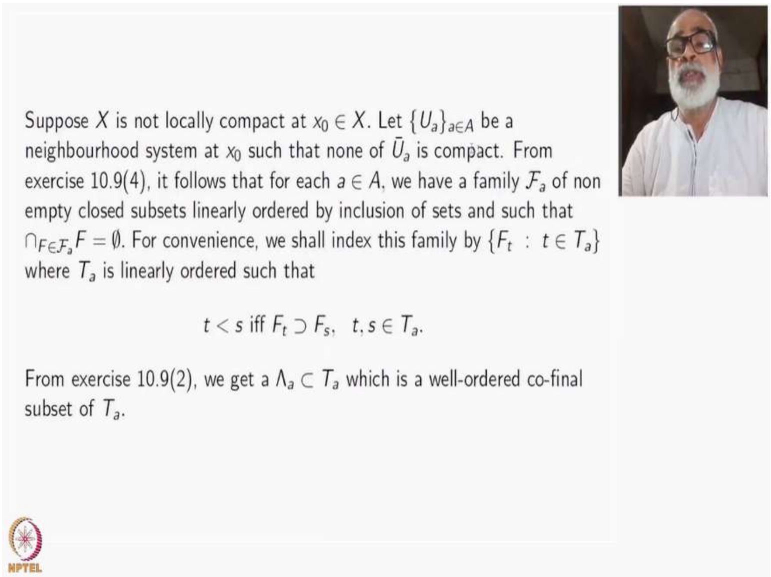

So, we add one extra point to each Λ_a. You can call that extra point ∞_a or something else. But I will call it Λ_a itself. This is a very convenient notation, take Λ'_a equal to Λ_a disjoint union singleton $\{\Lambda_a\}$.

Just one extra element, that extra element itself is Λ_a. Now, you extend the linear order on Λ_1 to entire of Λ'_a by just declaring Λ_a as the maximal element. A unique maximum element actually. Λ'_a is bigger than every element in Λ_a.

Now you take the order topology on Λ'_a.

Because there is a maximal element, this space will be a compact Hausdorff space. It is always Hausdorff space the topology coming from a well order. The compactness comes because of this extra point that you have taken. (It is like a one point compactification).

Now, put Y equal to the disjoint union of all these Λ_a's where a ranges over A. Starting with a special neighbourhood system at x_0, which is indexed by A, we have now come to the space Y which is a disjoint union of all these compact spaces Λ'_a's. Note that this Y itself may not be compact. It has disjoint union topology, where, on each Λ'_a we have taken have the order topology.

Having defined the space Y, now I construct its quotient space Z by identifying all these extra points Λ_a to a single point: Λ_a will be related to Λ_b, for all a, b inside A. And y related to y for all other point so Y. The quotient set for this equivalence relation will be denoted by Z and the class of all Λ_a's will be denoted by z_0. Only the extra points that you have taken Λ_a, all of them are identified together to single point, z_0 in Z. Let q from Y to Z denote the quotient map.

This quotient map has the property that over z_0, the fibre, i.e., $q^{-1}(z_0)$ is the collection of all Λ_a, $a \in A$. And the inverse image of every other point is just one single point. So this is the quotient map. That means of course, we are taking the quotient topology on Z. That is all.

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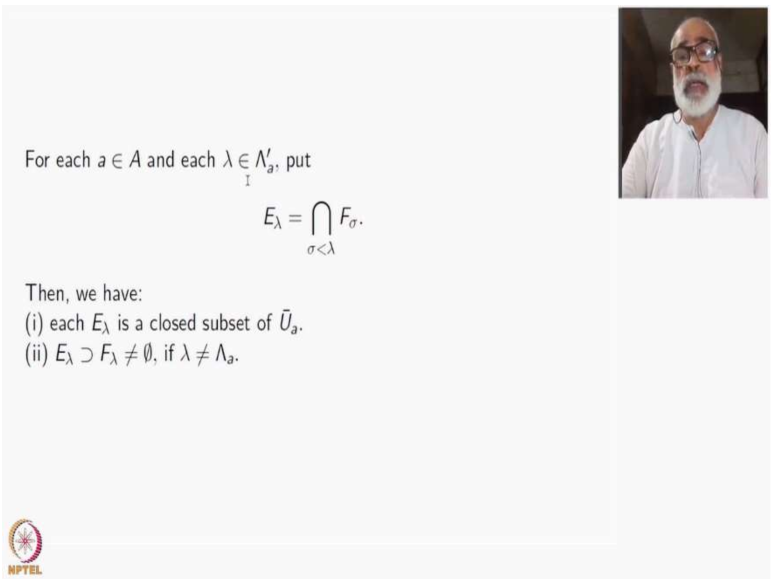

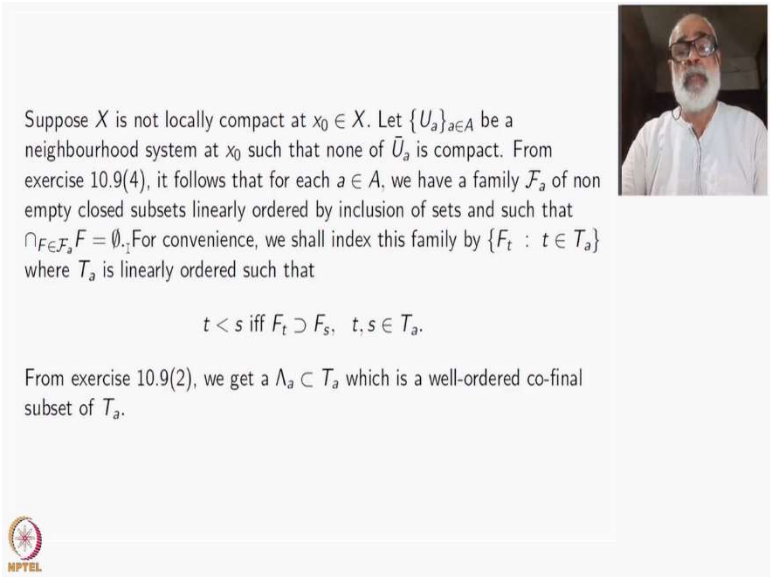

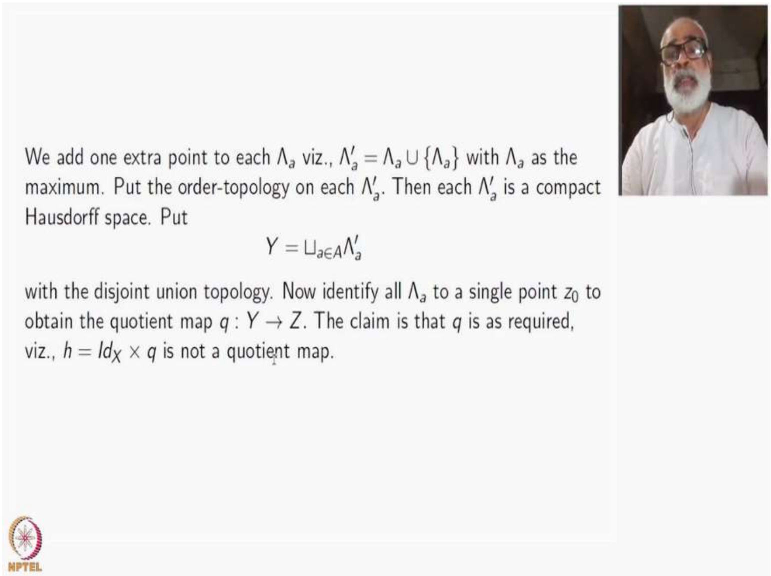

We claim that q is as required quotient map, i.e., such that identity of X cross q is not a quotient map.

So here is a preparatory result. For each $a \in A$, and for each $\lambda \in \Lambda'_a$, put E_λ equal to the intersection of all F_σ where σ is less than λ. So, if this λ was Λ_a, then what is E? This intersection will be the intersection of all F_t's, where t belongs to this Λ_a, and what was the assumption? Remember, the assumption was that this intersection is empty for each a.

If I take λ to be anything smaller, then what happen? Because it is a decreasing sequence ... In any case,

(i) first of all, being the intersection of closed subsets each E_λ is a closed subset of $\overline{U_a}$. Remember these are all subsets of $\overline{U_a}$. E_λ is contained inside F_λ and F_λ is not empty, if λ is not the whole of Λ_a.

(ii) Each E_λ contains F_λ here already, because E_λ is what? Intersection of all F_σ's where each F_σ contains F_λ (reversed inclusion order), and F_λ is non empty, every member is non empty. So, if this λ is not the last element, then this is non empty. λ is the last element then of course this is empty. So, that is all elementary observations.

Now, we want to show that the map h:=identity cross q is not a quotient map.

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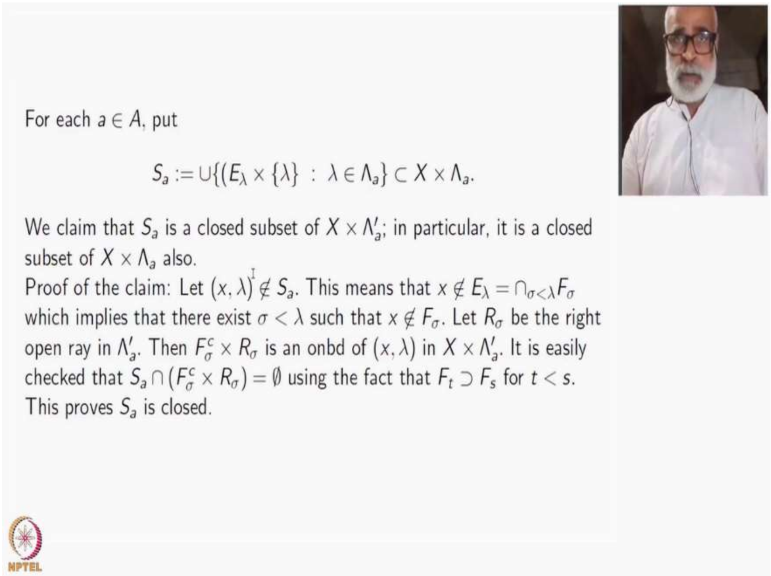

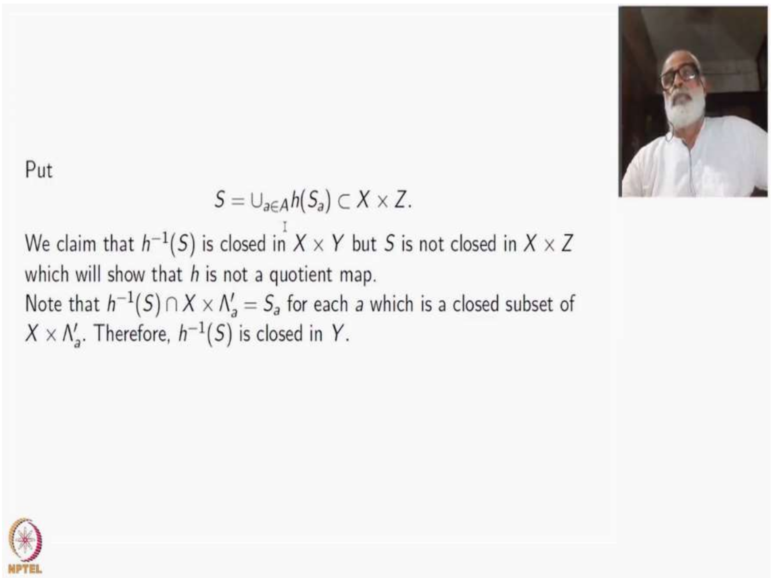

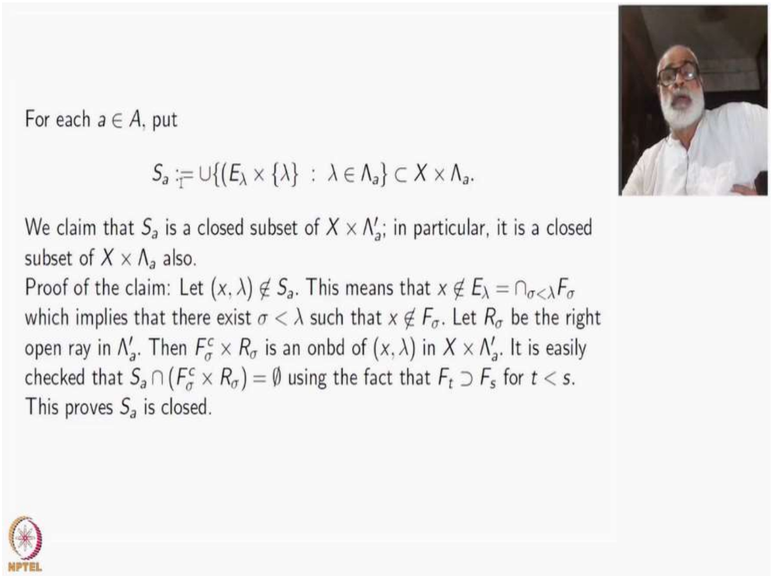

What we do? We take a closed subset of the total space Y which is the inverse image of some set, that some subset below is not closed. So, that is what we have to produce.

For each $a \in A$, put S_a equal to the union of all $E_\lambda \times \{\lambda\}$, where λ runs over Λ_a. This S_a is a subset of the product space $X \times \Lambda_a$. I have not taken the last maximal element here.

We claim that S_a is closed subset of $X \times \Lambda'_a$ itself, though it is a subset of $X \times \Lambda_a$. In particular, it would follow that it is closed in $X \times \Lambda_a$ also.

See, it is easy to see that each $E_\lambda \times \{\lambda\}$ is a closed subset but why the union, an arbitrary union over all the λ should be closed? So, how do you prove that? Take a point (x, λ) which is not in S_a. Want to show something is closed, show that its complement is open).

So, start with the point (x, λ) which is not in S_a. What is the meaning of this is not in any of these sets, is the union of these things. So, this means that the first coordinate x is not inside E_λ. (Note that if $\lambda = \Lambda_a$, then E_λ is empty and this is vacuously true.) (If x is inside E_λ for one of the λ's then this would imply that (x, λ) is in $E_\lambda \times \{\lambda\}$ contained in S_a.) But E_λ is what? Intersection of all F_σ, where σ is less than λ.

If something is not in the intersection, it means that there exist some σ less than λ such that x is not in F_σ.

Let R_σ be the right open ray inside Λ'_a. Remember these are all well ordered subsets, right rays etc all make sense.

So, once you have got some sigma here, R_σ means what? All those λ's which are bigger than σ, including your Λ_a. R_σ is an open neighbourhood of λ and also of Λ_a. Then $F_\sigma^c \times R_\sigma$

contains x because x is not in F_σ, and R_σ is an open subset which contains λ) is a neighbourhood of (x, λ) in $X \times \Lambda'_a$.

It is easily checked that S_a intersection this open subset is empty. What does that mean? That this neighbourhood is contained in the complement of S_a. So, what I have shown here is that (x, λ) belonging to S_a has an open neighbourhood which does not intersect S_a at all, open neighbourhood. That means, this complement of S_a is open. Over.

So, why this S_a is not empty? For this you have to use the elementary fact that each $\mathcal{F}_a$ has a least element that is the biggest subset F_0 which is non empty.

Then for any x in F_0, we have $(x, 0+1)$ will be in S_a. So, we have got a set here which is a closed subset of $X \times \Lambda'_a$. So, in particular this is a closed subset of $X \times \Lambda_a$ also, because it is a subset of $X \times \Lambda_a$ to begin with.

So, Y has this disjoint union of closed subsets that is a closed subset, in the disjoint in topology, there is no problem. But S is its image $h(S_a)$ as a ring over h, is a same thing h of all these disjoint unions, that is subset of $X \times Z$. Because first coordinate, I have taken as X here. So, we claim that $h^{-1}(S)$ is a closed subset of $X \times Y$, which is nothing but I just told you disjoint union of all these sorry here, all these S_a, which say, well, subset of X cross this Y here, not $X \times Y$.

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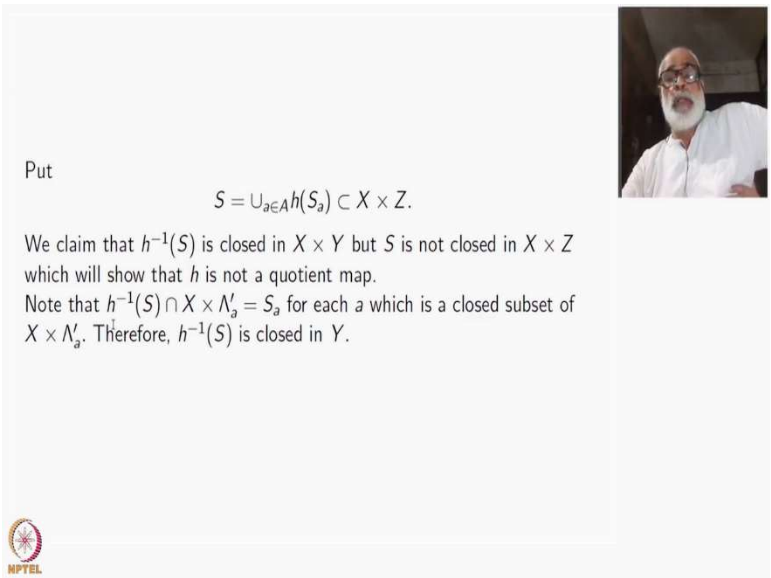

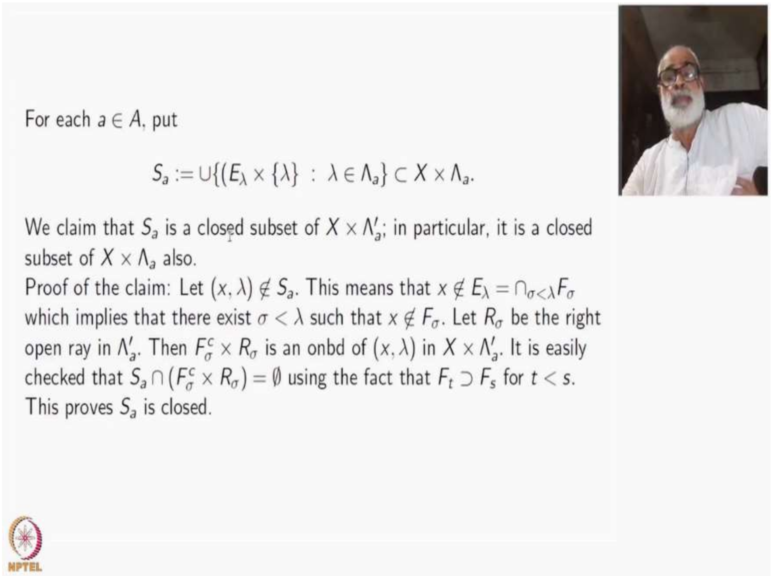

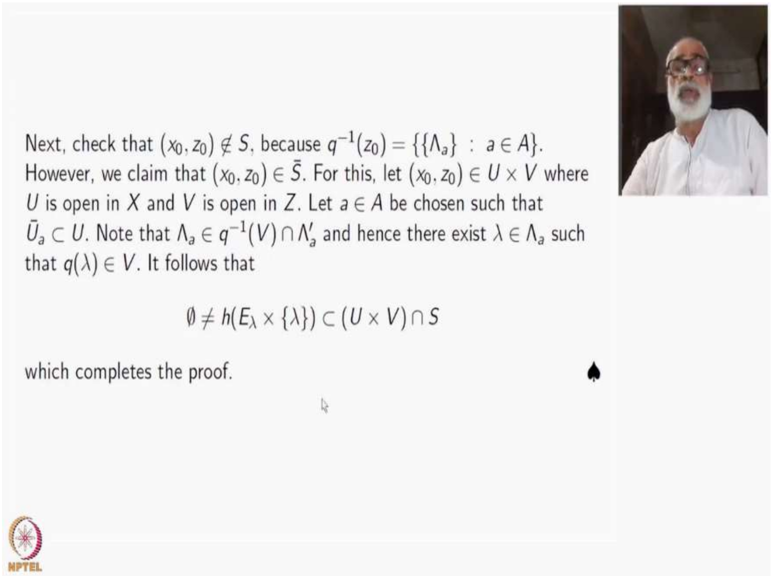

Now, take S to be the union of all $h(S_a)$ where a runs over A. S is a subset of $X \times Z$.

The union of all S_a as a runs over A is a subset of $X \times Y$, which is a disjoint union of $X \times \Lambda'_a$ subset of $X \times Y$ in the disjoint union topology. But now S is its image under h, and $h^{-1}(S)$ is precisely this union which is a closed subset.

All that I need to show now is that S is not closed in $X \times Z$.

For this we want to first check that (x_0, z_0) is not in $h(S)$. That follows because $q^{-1}(z_0)$ is the set of all $\{\Lambda_a\}$, $a \in A$.

Secondly, we claim that (x_0, z_0) is in the closure of $h(S)$. Something is in the closure and not inside S, just means that it is not a closed set, that is all.

So, for this start with a neighbourhood $U \times V$ of (x_0, z_0), where U is open in X and V is open in Z. I want to show that $U \times V$ intersects S. Let $a \in A$ be such that $\overline{U_a}$ is contained in U. So, what I am using here? That this family $\{\overline{U_a}\}$ form a neighbourhood system at x_0.

So, if U is a neighbourhood of x_0, there will be some a such that $\overline{U_a}$ is contained inside U. These are a system of closed neighbourhood none of them is compact). That is how we have chosen.

Note that this Λ_a is in $q^{-1}(V \cap \Lambda'_a)$, the top elements are mapped onto z_0 by q and z_0 is in V. So, $q^{-1}(V)$ will have this Λ_a. Therefore $q^{-1}(V) \cap \Lambda'_a$ is a neighbourhood of Λ_a in the topology of Λ'_a. And hence, there exists lambda in Λ_a such that R_λ is contained in this neighbourhood. None of the open right rays R_λ is empty because Λ_a itself has no maximal elements (that would have meant that intersection of all members of $\mathcal{F}_a$ is non empty). That means for some $\sigma \in \Lambda_a$, we have $q(\sigma)$ belongs to V. It follows that $h(E_\sigma \times \{\sigma\})$, (come down to $X \times Z$ under the quotient map,) is contained inside $(U \times V) \cap S$. See this h just identifies all these Λ'_a, with z_0, and when λ are smaller it is keeping them the same elements here, there is the identity map. So, that will be inside $(U \times V) \cap S$. Which completes the proof.

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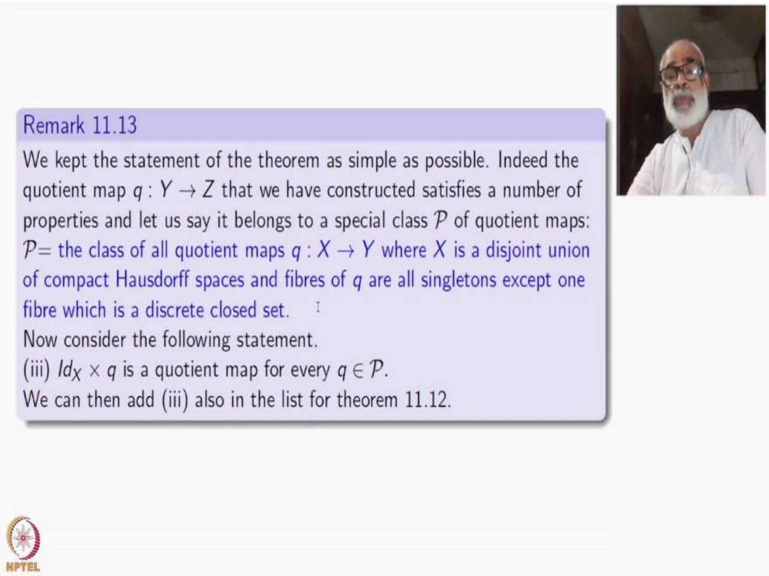

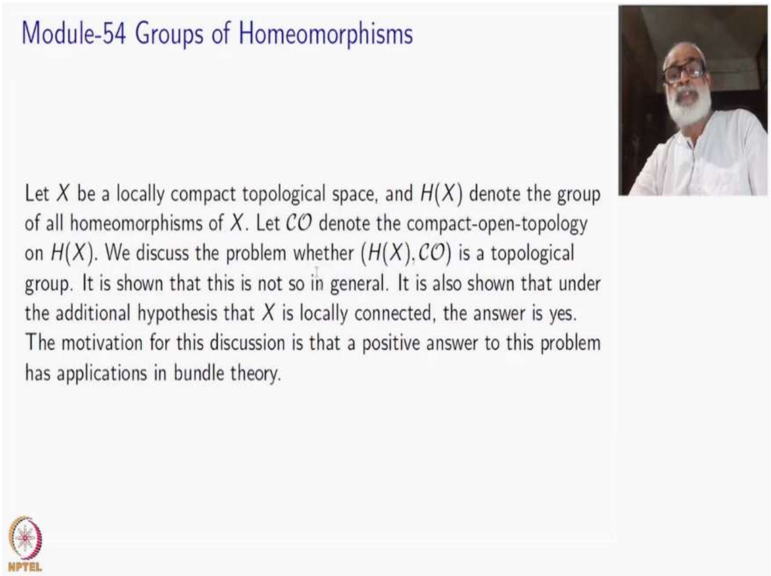

So, I will just make this remark: We kept the statement of the theorem as simple as possible unlike the original statement given in Michael's paper. Indeed, a quotient map that we have constructed satisfies a number of properties. Let us say it belongs to a special class P of quotient maps. This class P is the class of all quotient maps from X to Y, where X is a disjoint union of compact Hausdroff spaces and fibres of q are all singletons accept one fibre which is a discrete closed set. Actually this X is our Y and Y is Z, I have deliberately changed them because you should be able to do that on your own also.

What we have seen is that the quotient maps are very peculiar. Only inverse image of a single point one having too many points, for rest of the points inverse image consists only one point, even such maps such maps can be badly behaved! Identity cross such a map is not a quotient map, where you take then X to be non locally compact. That is the statement that we have actually proved. But we do not want to insist on that one. So, we can add these 3 also in the theorem, $1, 2, 3$ are all equivalent. This will be much weaker statement than 2. I mean, looking, because it is a subclass. That is all. Thank you. Next time we will do something else.