An Introduction to Point - Set - Topology (Part II)
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Lecture No. 52
The Exponential Correspondence

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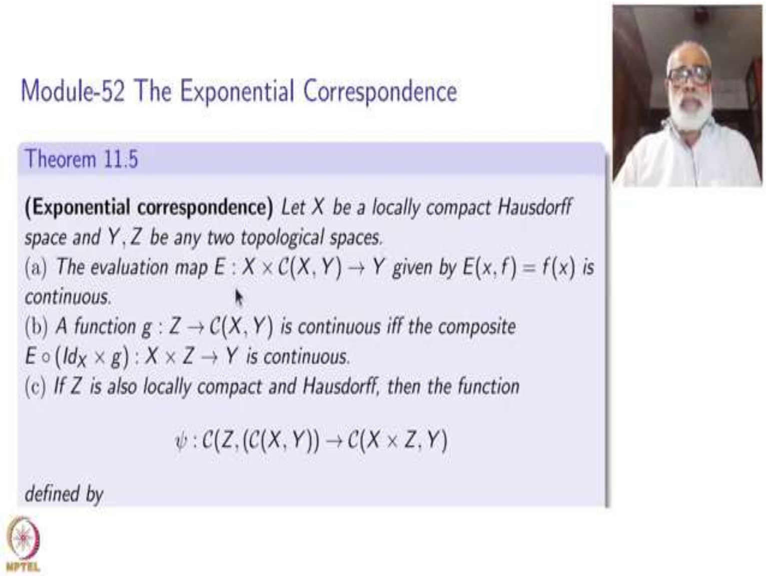

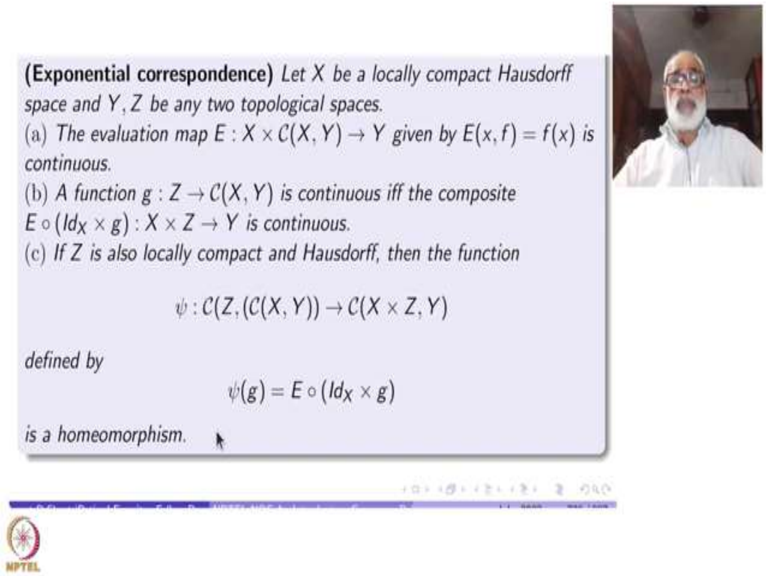

Hello, welcome to module 52 of NPTEL, NOC, an introductory course on Point Set Topology, Part II. So we shall continue our study of function spaces. Today the topic is Exponential Correspondence. So let me start with a theorem because I have already introduced all the relevant notation, I hope you have done your homework for getting familiar with this notation.

Start with a locally compact Hausdorff space X, Y and Z are any two topological spaces.

(a) The evaluation map E from $X \times \mathcal{C}(X, Y)$ to Y, given by $E(x, f) = f(x)$ is continuous.

(This is the first statement of (a).)

(b) A function g from any topological space, (there may be some other topological space here) Z to $\mathcal{C}(X, Y)$ is continuous if and only if $E \circ (Id_X \times g)$ from $X \times Z$ to Y is continuous.

(So this gives you a criterion for determining when functions into $\mathcal{C}(X, Y)$ are continuous.

(c) If Z is also locally compact and Hausdorff, (locally compact Hausdorffness is always a standing assumption on X, now if Z is also locally compact and Hausdorff), then the function from ψ from $\mathcal{C}(Z; \mathcal{C}(X, Y))$ to $\mathcal{C}(XZ, Y)$, the exponential correspondence, (the function that we defined last time, I am repeating it here) namely, $\psi(g) = E \circ (Id_X \times g)$, this is a homeomorphism.

Notice that by this statement (b), given any g here, $\psi(g)$ is defined by this rule is continuous. `if' part you do not need. So the ψ, though it takes values actually in $Y^{(X \times Z)}$ in general, it is actually inside this smaller subspace of continuous functions. That is why I can write like this. The statement (c) is much more stronger, says that psi is actually a homeomorphism.

So let us go through the proofs of these things carefully one by one.

One point which you can remember is the following and how to remember. Whenever you take continuous functions from somewhere, the domain must be locally compact Hausdorff or locally compact regular, some local compactness has to be there. Right in the beginning we took function from X to Y; therefore, we assumed X is locally compact Hausdorff. There was no need to assume anything on the codomain.

Similarly, here in (b) there is no assumption on Z, the function is taking values inside $\mathcal{C}(X, Y)$. So before taking $\mathcal{C}(X, Y)$, I would like to ensure that X is locally compact Hausdorff. So that is the case when you come here to (c) because now you have to take $\mathcal{C}(X, Y)$ and then $\mathcal{C}(Z, \mathcal{C}(X, Y))$ so I am assuming Z is also locally compact Hausdorff.

X and Z are locally compact Hausdorff, the product will be locally compact Hausdorff, so there is no extra assumption. So still Y is a free topological space, no condition on Y. So, it is easy to remember where you put the hypothesis.

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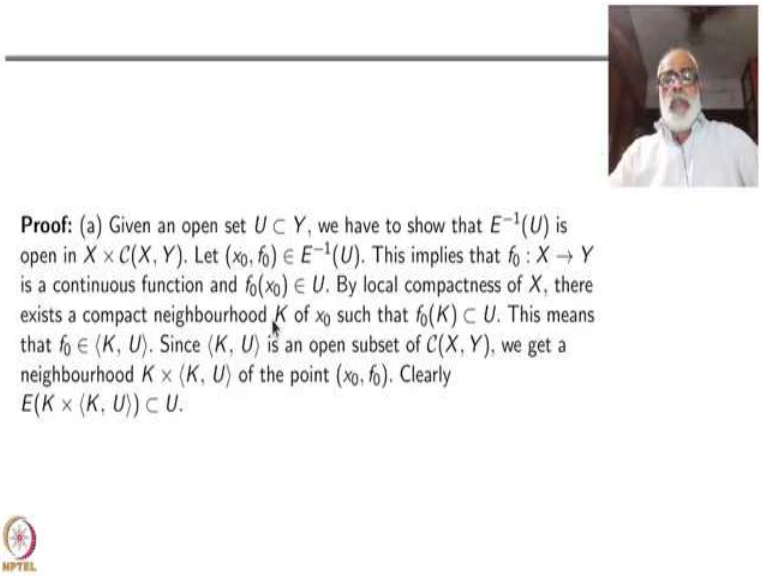

So, first the proof of (a). I want to prove that the evaluation map itself is continuous. Given an open set U in Y, we must show $E^{-1}(U)$ is open in $X \times \mathcal{C}(X,Y)$. So take a point here, (x_0, f_0) belonging to $E^{-1}(U)$ contained in $X \times \mathcal{C}(X,Y)$. What is the meaning of 'it belongs $E^{-1}(U)$'? This implies that f_0 is a continuous function from X to Y, and $f_0(x_0)$ is in U.

Now use local compactness of X, you can find a compact neighbourhood K of x_0 in X such that $f_0(K)$ is contained in U. (Actually you get an open set V such that x_0 in in V, $\overline{V}$ is compact and contained in $f_0^{-1}(U)$. I have combined all that and taken $K = \overline{V}$.)

This means that f_0 is in $\langle K, U\rangle$. Since $\langle K, U\rangle$ is an open subset of $\mathcal{C}(X,Y)$, we get a neighbourhood $K \times \langle K, U\rangle$, of (x_0, f_0) in the product topology of $X \times \mathcal{C}(X,Y)$.

Now clearly you take E of this neighbourhood, by the very definition of $\langle K, U\rangle$, is inside U. that is the meaning. So we have found out a neighbourhood of the point which is completely taken inside U by E; that is the meaning of that E is continuous.

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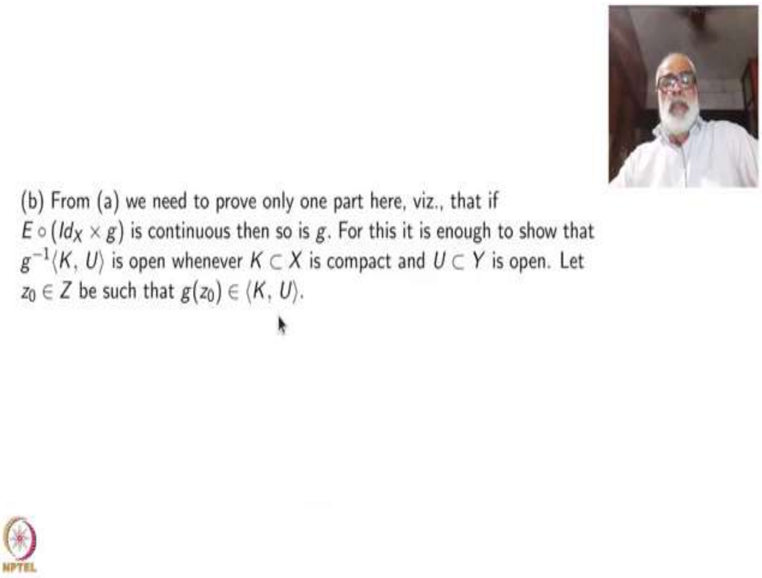

Now, let us prove (b). Now we start with an arbitrary space Z and a continuous function g, identity of X cross g is continuous, E is continuous because we have proved it in (a), so composite is continuous. So one way is obvious. Now, what we want to do is assume that this composite is continuous. Then you have to prove that this g is continuous.) See composite of two functions can be continuous without either of them being continuous. You must know such examples. If both f and g are continuous then the composite is continuous, that is a standard result that we have proved, but it is possible to have neither of them continuous but the composite is continuous.) Here we have to prove this nontrivial result, assuming this composite is continuous, you have to prove that g is continuous.

For this it is enough to show that g inverse of these subbasic open sets in $\mathcal{C}(X, Y)$ are open.

So this is standard method we have been following. We should have proven that inverse image of every open set is open, you can just take only subbasic open sets. What are the subbasic open sets here? K compact U open, then you take $\langle K, U \rangle$. So start with a point z_0 belonging to Z such that $g(z_0)$ is in $\langle K, U \rangle$. So $g(z_0)$ is a continous function, remember that now. Because g is a function of Z to $\mathcal{C}(X, Y)$.

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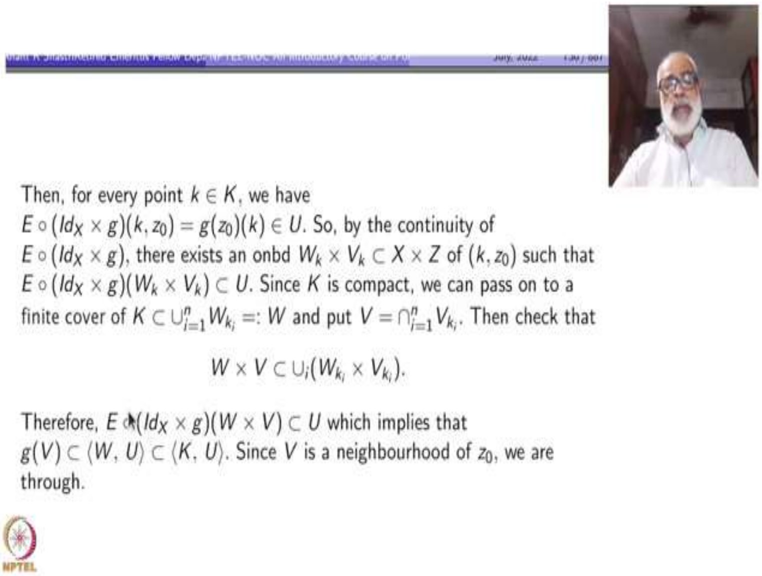

Then for every point $k \in K$, look at this composition, E composite identity of X cross g operating upon (k, z_0) is nothing but $E(k, g(z_0)) = g(z_0)(k)$ the first coordinate remains k and then evaluation. This is inside U. So, by continuity of E composite identity of X cross g, that is the hypothesis now, there exists a neighbourhood, $W_k \times V_k$ inside $X \times Z$ of this pair (k, z_0) such that E composite identity of X cross g, operating on this product neighbourhood goes inside U. As k varies over K, we get an open cover $\{W_k\}$ for K and since K is compact, we can pass on to a finite subcover of K, say K contained in the union of W_{k_i}, for $i = 1$ to n. I will call this union W. What we have done? For each $k \in K$ we have got a neighbourhood W_k, of k, so these $\{W_k\}$ is an open cover the whole of K which is compact. Therefore, I got a finite subcover here, which I am denoting by $W_{k_1}, W_{k_2}, \ldots, W_{k_n}$. On the other hand I take V to be intersection of the corresponding V_{k_i}'s. Then $W \times V$ (this product of union and intersection) that will be contained inside the union of $W_{k_i} \times V_{k_i}$ where i ranges from 1 to n. Any point (x, z) belongs to the LHS means x is in some W_{k_i} but z is in V_{k_j} for all j. Therefore, (x, z) is in $W_{k_i} \times V_{k_i}$ and hence it is in the union.

Therefore, if you apply E composite identity of X cross g on this one, that will be inside U now, because the same is true for each set on the RHS. This in turn implies that $g(V)$ is contained in $\langle W, U\rangle$. But $\langle W, U\rangle$ is contained inside $\langle K, U\rangle$, because K is contained inside U, and W is larger. Anything which brings the entire of W inside U must be also bringing K inside U.

Since V is a neighborhood of z_0 we are done. We have found out a neighborhood V of z_0, so that $g(V)$ is inside this sub-basic open set, that is what we have started with.

So we have to go through these steps to prove the converse part here. Namely, just assuming that this composite is continuous, we proved that g is continuous, so that is part (b).

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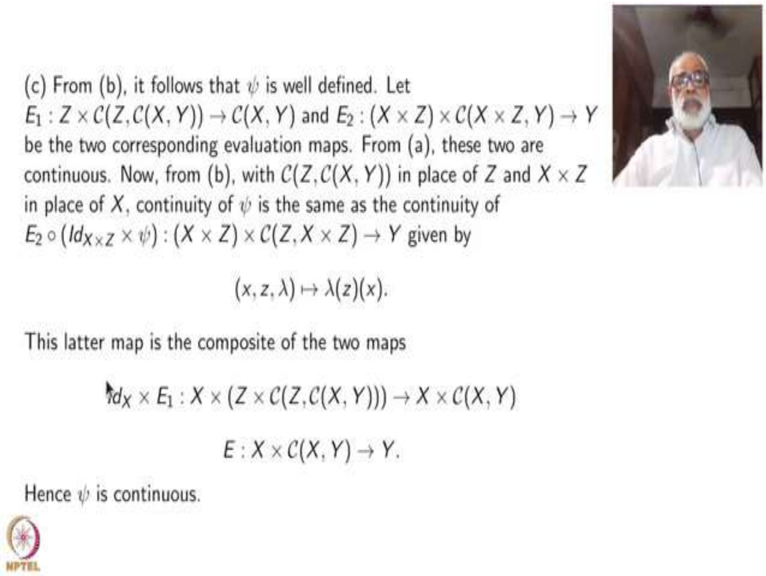

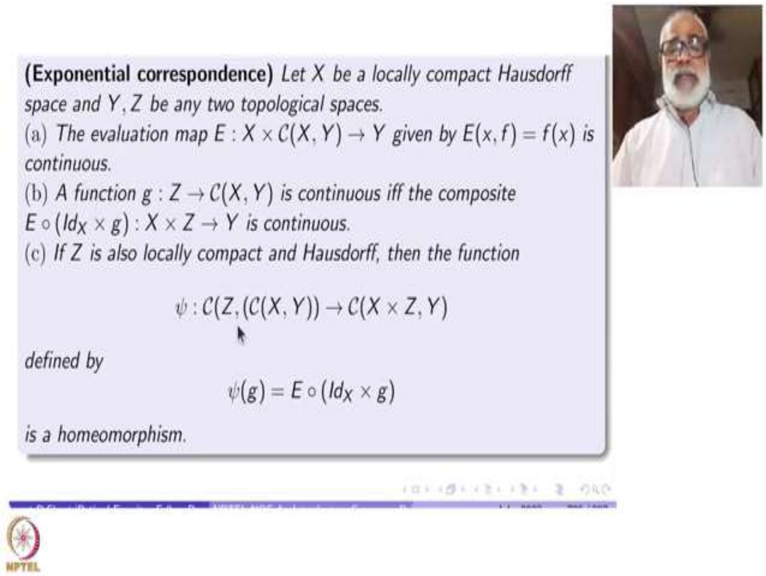

Now we will do the last part (c), namely, exponential correspondence ψ is a homeomorphism. As pointed out before, first of all, because of (b) ψ is well defined. I repeat this is what I told you right in the statement here, because of (b) this makes sense, that this is function from here to here. Otherwise, I am not at liberty in choosing the definition of ψ. ψ has to be defined like this at the set theoretic level itself. I have to only verify that the codomain is correct when g is continuous.

So that is the first remark again. This is repeated here that is all. So ψ is well defined.

I will have two more evaluation maps here, remember this E was from $X \times \mathcal{C}(X,Y)$ to Y. Similarly, I have this E_1 from $Z \times \mathcal{C}(Z,\mathcal{C}(X,Y))$ to $\mathcal{C}(X,Y)$ and E_2 from $(X \times Z) \times \mathcal{C}(X \times Z, Y)$ to Y. So each time, in the domain we have product of a space with the space of continuous functions form the same space into another space. And the codomain is always that other space. That is the pattern for domains and codomains of evaluation maps.

So we have to deal with three such evaluation maps. Because we are assuming that Z locally compact and Hausdorff, the statement (a) is valid for all of them. That means that these evaluation maps are continuous. Now we apply (b) by taking $\mathcal{C}(Z,\mathcal{C}(X,Y))$ in place of Z and $X \times Z$ in place of X, for the continuity of ψ. What is the domain of ψ? It is $\mathcal{C}(Z,\mathcal{C}(X,Y))$. What is the codomain? It is $\mathcal{C}(X \times Z, Y)$. So that is why I am taking $X \times Z$ in place of X. So, continuity of ψ is the same thing as continuity of this E_2 composite ((identity of X cross Z) cross ψ).

If I prove that this composite is continuous, then ψ will be continuous. (It is possible to prove continuity of ψ directly by elementary methods, but I find hat will have cumbersome notation. So I find this way is easier to state and observe, once you have done the ground work here, namely (b), you can keep using (b) again and again.)

So I will show you E_2 composite, with identity of X cross Z cross ψ, is continuous. The domain is $X \times Z \times \mathcal{C}(Z,\mathcal{C}(X,Y))$ and the codomain is Y.

From $X \times Z$ we have the identity to $X \times Z$, and from $\mathcal{C}(Z,\mathcal{C}(X,Y))$ we have ψ to $\mathcal{C}(X \times Z, Y)$. So given $x \in X, z \in Z$ and lambda in $\mathcal{C}(Z,\mathcal{C}(X,Y))$, $\lambda(z)(x)$ is precisely equal to $E_2(x,z,\lambda)$. We can rewrite this as a composite two different maps, viz., identity of $X \times E_1$ from $X \times Z \times \mathcal{C}(Z,\mathcal{C}(X,Y))$ to $\mathcal{C}(X \times Y)$, first followed by the evaluation map E.

See under the first map, (x,λ,z), this x remains as it is, E_1 is the evaluation map operatiing on (λ, z) and so that becomes $\lambda(z)$, which is an element of $\mathcal{C}(X,Y)$. Then you apply again the evaluation map E here on $(x,\lambda(z))$. This E is from $X \times \mathcal{C}(X,Y)$ to Y, that is what we have started with. So that gives you $\lambda(z)(x)$.

But now these two are continuous, therefore, this composite is continuous. So that establishes that ψ is continuous.

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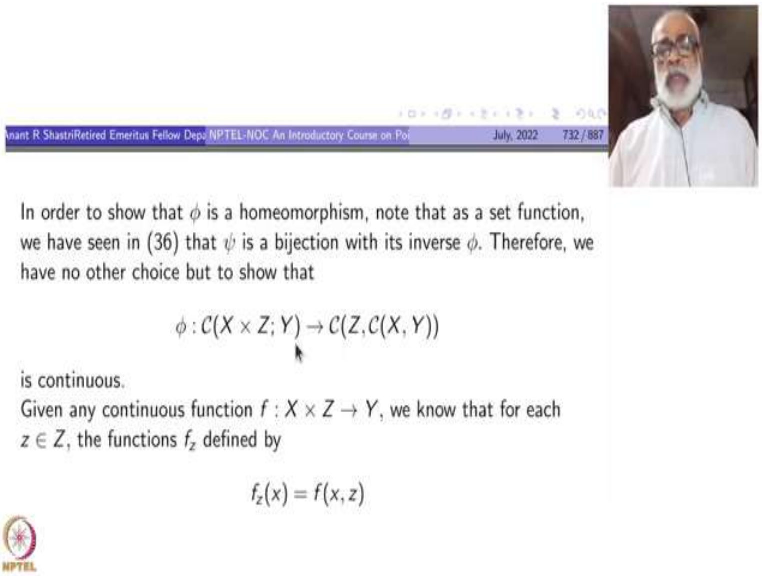

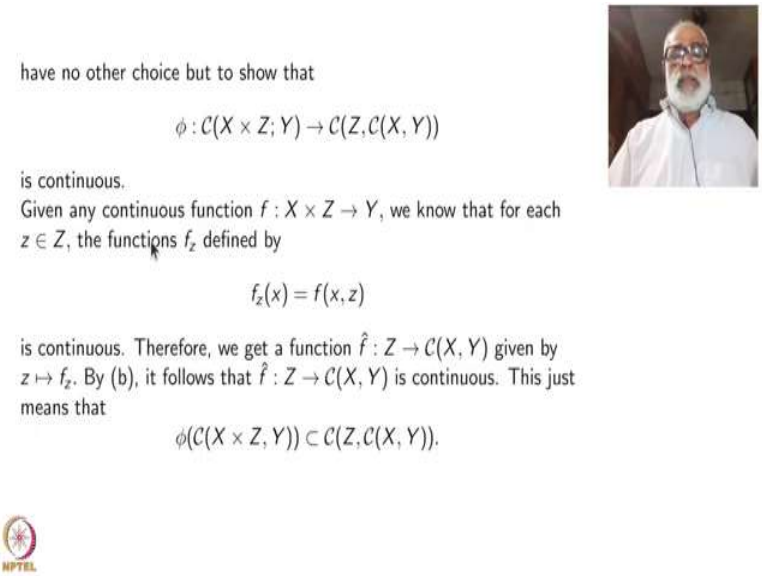

Exactly similarly, you can show that ψ^{-1}, namely, ϕ is continuous. In fact, we want to show that ϕ is a homeomorphism, same thing as ψ is a homeomorphism. Note that as a set function, we have seen that this ψ is a bijection and its inverse is ϕ. Therefore, we have no other choice on the subspaces also they are inverses of each other. On each smaller subset the inverse will be corresponding restrictions; that is all. So I have to take the same ϕ, and show that this is continuous now. Then it will prove automatically that both ϕ and ψ are homeomorphisms. So let us prove that ϕ is also continuous. So proof is even simpler here, but anyway you have to go through these steps.

Namely, given any continuous function f from $X \times Z$ to Y, we know that for each z in Z the partial functions f_z, namely, you are fixing one coordinate z and taking x going to $f(x, z)$.

Those things are continuous that you know already. Joint continuity implies partial continuity. Therefore, we get a function $\hat{f}$ viz., $\hat{f}(z) = f_z$.

This function itself is continuous. Why? Now you apply (b). So you have to see that Z to $\mathcal{C}(X, Y)$ continuous, you have to take $X \times Z$ to that evaluation map composite with identity cross this one. So if you apply that it will follow that $\hat{f}$ is continuous, because finally, what you get is $f(x, z)$, when you evaluate $\hat{f}$.

This means that, first of all, that under this ϕ, $\mathcal{C}(X \times Z, Y)$ goes inside $\mathcal{C}(Z, \mathcal{C}(X, Y))$. Because it goes to $\hat{f}$ and $\hat{f}$ is a continuous function from Z to $\mathcal{C}(X, Y)$. So this is just a justification that ϕ has the correct domain and codomain. Now we have to show that ϕ is continuous. That part is still there.

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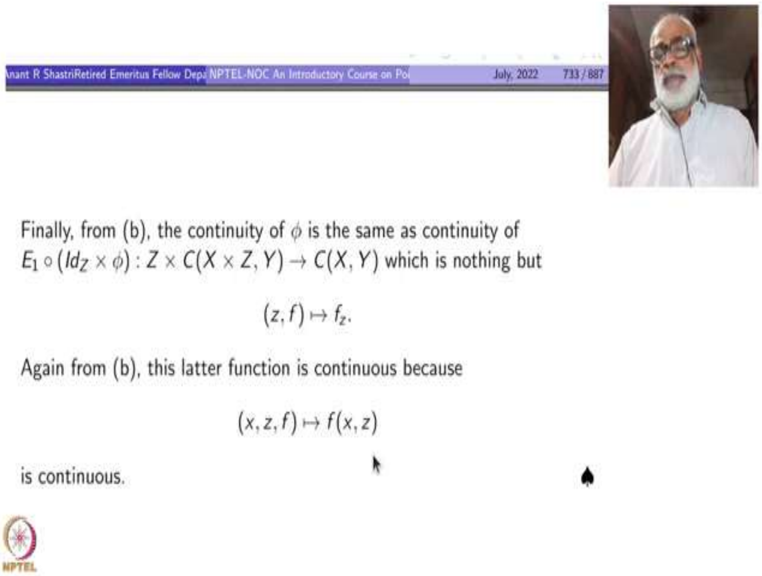

From (b), the continuity of ϕ is the same thing as continuity of E_1 composite identity of Z cross ϕ. So what is that function? That is (z, f) going to $f(z)$. Again by (b) this latter function is continuous because when you evaluate it on x, we get (x, z, f) going to $f(x, z)$ and this is continuous. Because f is continuous, this is an evaluation map from $X \times Z$ to whatever, C of $X \times Z, Y$. All right!

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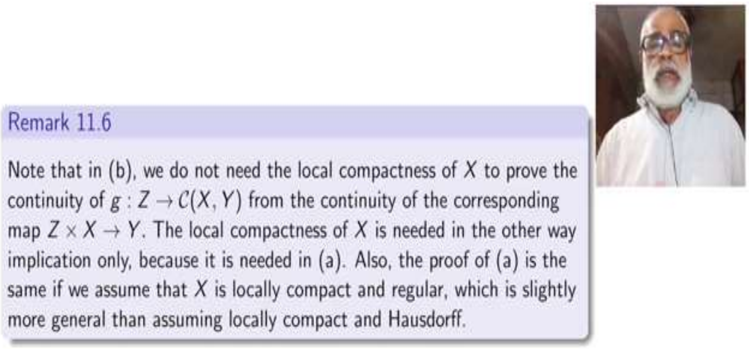

So we have established exponential correspondence in a very strong sense, viz., Homeomorphism types of $Y^{X \times Z}$ and $(Y^X)^Z$, if you interpret them as corresponding subset of continuous functions. You can remember this easily. The condition is that whatever space is taken as an exponent must be locally compact and Hausdorff.

Here is a remark. Note that in (b), we do not need the local compactness of X to prove the continuity of g from Z to $\mathcal{C}(X, Y)$. This is only needed in the proof of (a), the local compactness of X is needed in the other way round implication only, because it is needed in (a), if you want to apply other way round, then you have to use (a). Also, the proof of (a) is the same if we assume X is locally compact and regular, (instead of Hausdorffness), which is slightly more general than assuming locally compact Hausdorff.

Because we have seen that locally compact and Hausdorff implies regularity, but the other way may not be true. So we can also do these result with locally compact regularity. All that I have used here is that points of X have arbitrary small neighbourhoods with compact closures, i.e., The set of compact neighborhoods of a point forming a fundamental system at each point.

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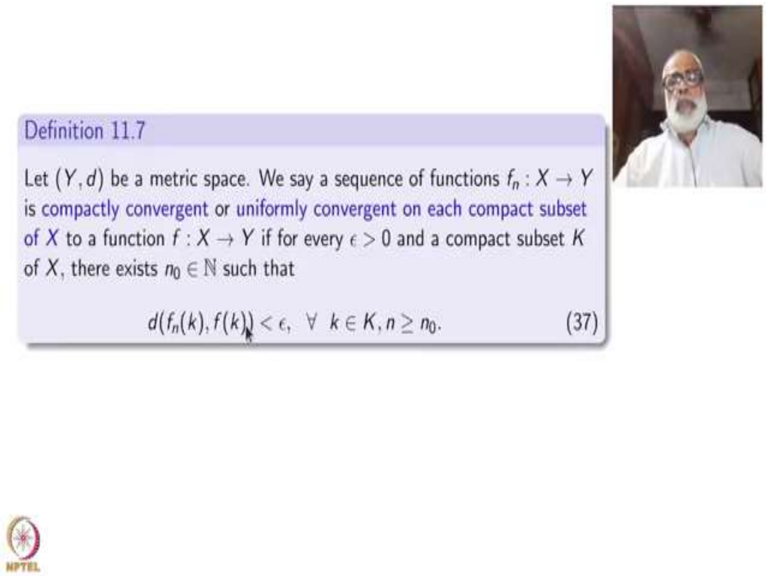

So now, we shall do one more justification for introducing this compact open topology. All right. Let (Y, d) be a metric space. We say a sequence $\{f_n\}$ from X to Y is compactly convergent or uniformly convergent on each compact subsets of X, (this is a longer wording, whereas the first one is a neat wording, both are used commonly) to a function f, if what happens, I mean, when is this happening namely, if for every ϵ positive and a compact subset K, there exists n_0 such that distance between $f_n(k)$ and $f(k)$ is less than ϵ for every k inside K and for every n bigger than n_0. If this happens for each k and then this n_0 depending upon k, that will be just point-wise convergence. So if the same n_0 works for all the points of k that is uniformly on k. K is compact that is why it is called compactly convergent. So this is not my definition. This is the standard definition. I am just recalling it.

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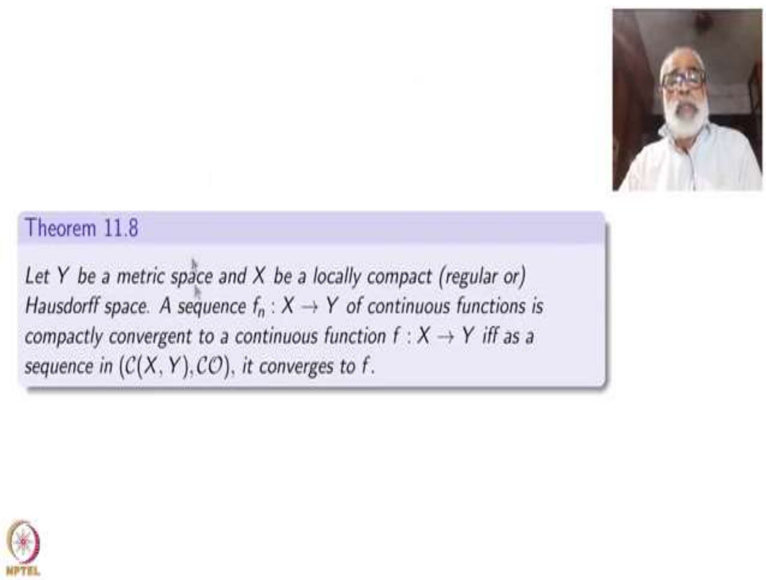

Now we come to something, let Y be a metric space, X be any locally compact Hausdorff or a locally compact regular space. One of them. A sequence f_n from X to Y of continuous functions is compactly convergent to a continuous function f from X to Y if and only if as a sequence in $(\mathcal{C}(X,Y),\mathcal{CO})$, i.e., with the compact open topology, it converges to f.

Here the hypothesis is that the sequence is of continuous functions and the limit is also continuous. Then compactly convergence implies the usual convergence in this topology, that is what the final conclusion here. We are not proving that that the limit function f is continuous. Indeed that is true, you know already in the case of metric spaces.

If it is compactly convergent sequence of continuous function, then f is continuous. But we are not proving that statement for compact open topology. There we are assuming this one. The statement is that, namely, f_n converges to f, each f_n is continuous, f is continuous, the convergence can be in the general sense of metric spaces. Here you can say that it is in terms of compact open topology, that is the whole idea. The statement is clear, I hope.

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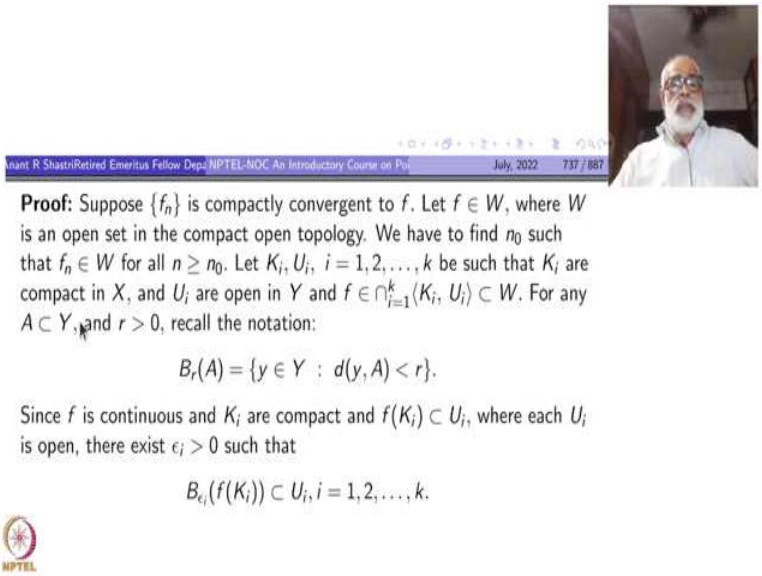

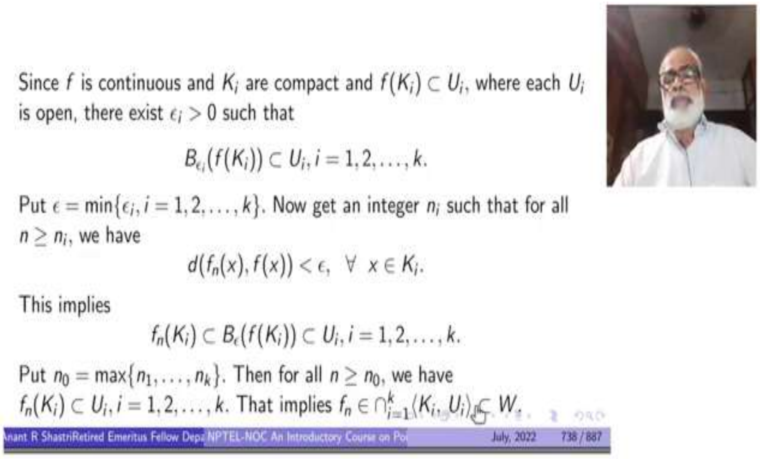

Now let us work it out. It takes a little bit of time, but this is routine. There is absolutely no new ideas here. All these things are standard methods in analysis. Suppose f_n is compactly convergent to f. Let f belong to W, where W is an open subset of $\mathcal{C}(X,Y)$ in compact open topology. We have to show that, there is some n_0 such that f_n belongs to W for all n bigger than n_0. So this will prove that f_n converges to f in the compact open topology.

So one way implication will be done first. Let K_i, U_i for $i = 1$ to k be such that K_i's are compact and U_i's are open and f is in the intersection of $\langle K_i, U_i \rangle$, the intersection itself contained in W. This is because these things make a subbase. So, whenever you have an open subset and a point in it, you can always get a member of the base as above, viz., intersection of finitely many members of the subbase.

Now for any A, any subset of this metric space Y and r positive, we have the standard notation, long back we have used this one, viz., $B_r(A)$ is the set of all y inside Y such that the distance between y and A is less than r. It is just the union of all open balls of radius r with center in A.

Now if f is a continuous function and K_i's are compact, $f(K_i)$'s are compact and contained in U_i where each U_i is open. Therefore, you can find an ϵ_i positive such that this open neighborhood $B_{\epsilon_i}(f(K_i))$ is contained inside U_i. All that you have to do is take ϵ_i to be distance between $f(K_i)$, which is compact and complement of U_i, which is closed. So if you have done it for all $i = 1$ to k, you can take this ϵ to the minimum of them. Once you have an ϵ you get an integer n_i, such that for all n bigger than n_i, we have distance between $f_n(x)$ and

$f(x)$ is less than ϵ for every $x \in K_i$ for all i. This is the compact convergence of the sequence f_n to f.

For each K_i, which is compact I will get an n_i here. But what is the meaning of this one? This means that, you see, x is in K_i and so $f(x)$ will be inside $f(K_i)$ and ϵ is same ϵ, so if you take the ϵ neighborhood of this $f(K_i)$, this f_n of the entire K_i must be inside here. That is true for all $i = 1, \ldots, k$.

Now, if you take n_0 to be the maximum of these n_i, then all of them will be simultaneously true. Therefore, $f_n(K_i)$ will be contained inside U_i for all i, which just means that f_n is intersection of these $\langle K_i, U_i \rangle$, but that is inside W. So one way we have proved.

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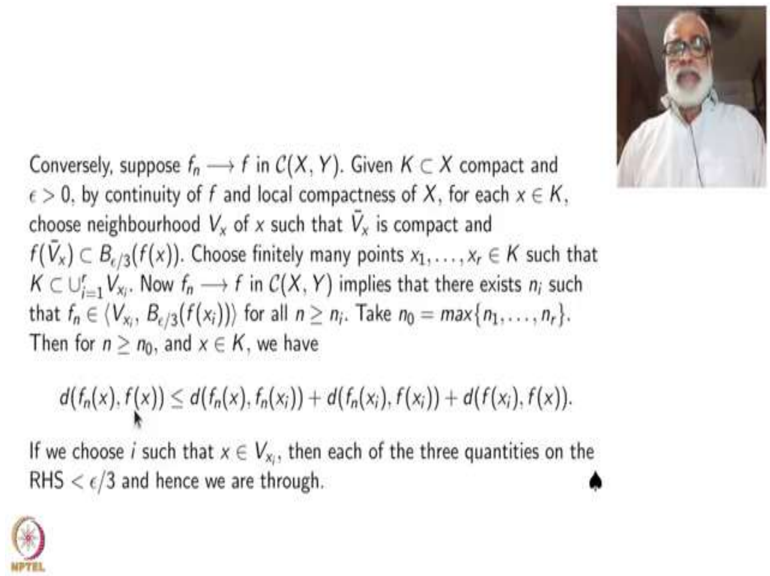

The converse, converse is more or less similar, but slightly different, now we will see. Suppose f_n converges to f in $\mathcal{C}(X, Y)$. Given K a compact subset K of X, and ϵ positive, by continuity of f and local compactness of X, (I have to use this one somehow) for each x belonging to K, choose neighborhood V_x of x such that $\overline{V_x}$ is compact and $f(\overline{V_x})$ is contained inside some neighborhood of $f(x)$, I will choose $B_{\epsilon/3}(f(x))$.

So that is the $\epsilon/3$ is an after thought, you know this kind of thing. If I have chosen the ϵ balls, finally, I will get 3ϵ balls. That is why I just start in the beginning itself $\epsilon/3$, that is all.)

So far we have used only local compactness and continuity of f (the sequence $\{f_n\}$ has not entered yet).

Now, choose finitely many points $x_1, x_2, \dots, x_r$ belonging to K such that K is contained in the union of $V_{x_i}\}$'s for $i = 1, 2, \dots, r$.

For each x inside K, I have got a neighbourhood V_x, with a certain property. They form an open cover for the compact set K. So I get a finite cover. This is again a routine thing.

Now f_n converges to f in $\mathcal{C}(X, Y)$ implies that for each of these i, there exists n_i such that f_n will be inside this open subset $(\overline{V_{x_i}}, B_{\epsilon/3}(f(x_i))$. This is an open subset in the compact open topology and f belongs to it.

Again take n_0 to be maximum of $n_1, \dots, n_r$. Then if n is bigger than n_0 and x is inside K, I am going to verify this uniform convergence part, viz., distance between $f_n(x)$ and $f(x)$ is less than or equal to $d(f_n(x), f_n(x_i)$ plus… first keep n the same, change x to x_i, next from $f_n(x_i)$ to $f(x_i)$ and finally from $f(x_i)$ to $f(x)$, each term contributing at most $\epsilon/3$ and hence the total is less than ϵ. So there are three quantities here.

If you choose i such that x is inside V_{x_i}, then each of these quantities on the right hand side will be less than $\epsilon/3$, because I have chosen i here. If they happens to be j, you have to put everywhere j, that is all. So they are $\epsilon/3$, so hence, we are through, some total is, I mean, this distance is less than ϵ.

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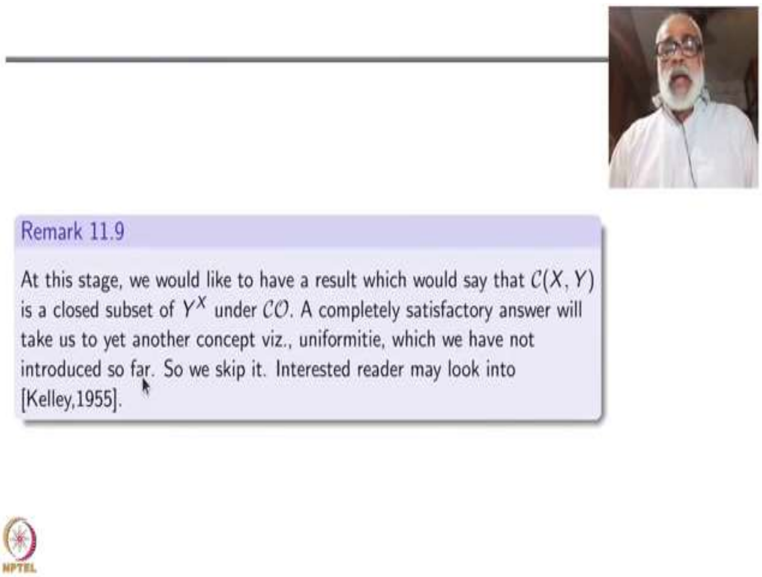

So I made a remark saying that if you have a sequence, which is convergent, in $\mathcal{C}(X, Y)$ a sequence of this one, which is suppose compactly convergent. So that will be a function, some function, would that be already continuous. See, I said I am not proving or I am not addressing this problem in this theorem, but that problem binds me, it can bother me and it does. As a function it is some function from X to Y.

So this question is the same as the following. Under the compact open topology, on Y^X itself, is the subset $\mathcal{C}(X, Y)$ closed in Y^X? This is the question.

A completely satisfactory answer will take us to yet another concept, which we have not introduced here. Now that concept has the name, `uniformities', which we have not introduced so far. So we shall skip it. There is no time for doing uniformities in this course. If you are interested, then we you can look into many books, special book for me is Kelly's book. Thank you. So next time we shall use this compact open topology to do some interesting applications. Thank you.